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Review article

SPECIFIC NUMERICAL PROPERTIES OF B-SPLINE IN FUNCTION APPROXIMATIONS

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Abstract

B-splines are a class of functions with interesting and numerically useful properties. Spline functions are piecewise polynomials connected by the $\Delta=\{a=x_1<x_2<...<x_n=b\}$ distribution on the $[a,b]$ segment in x_i nodes. *B*-spline is a combination of curves that pass through a certain number of points (control points) and form smooth curves. In this paper, we will consider *B*-splines as special partially nonnegative polynomials that disappear everywhere except in at several adjacent $[x_{i-1},x_i]$ intervals. From a numerical point of view, it is important to define *B*-splines through divided differences, with the possibility of computing higher-order *B*-spline recursively. *B*-spline approximations will be considered taking into account only the local behavior of the primitive function. We will use a numerically stable algorithm to efficiently calculate the estimate of the *B*-spline function. Some specific applications of *B*-spline calculated using the *Mathematica* program package and geometric interpretation of results are given.

Keywords: B-spline properties, Approximations via B-spline, Invers function formula, B-spline estimate, B-spline curve

1. Introduction

In general, the real function $f:[a,b]\to R$ is a piecewise polynomials of order k and degree $k-1$, if for every $i=0,\dots,n-1$, the restriction of the function f on subintervals (x_i,x_{i+1}) coincides with a polynomial $p_i(x)$ of degree less than or equal to $k-1$. In order to achieve injective mapping between f and the sequence $p_0(x),p_1(x),\dots,p_{n-1}(x)$, we define f in nodes $x_i(i=0,\dots,n-1)$ so that the function becomes continuous on the right. Some derivations of spline function may also be continuous, depending on whether successive nodes are different or not.

B-spline allow you to create and manage complex shapes and surfaces using a number of points.

B-spline of the order n are the basic functions of each spline function of the same order, defined on the same nodes, which means all possible spline functions can be built from a linear B-spline combination and there is only one unique combination for each spline function.

2. Definition of B-spline

Let $f_x : \mathrm{R} \to \mathrm{R}$ is function defined by

$$f_x(s) = (s-x)_+ = \max\left(s-x, 0\right) = \begin{cases} s-x, & \text{for } s > x \\ 0, & \text{for } s \,,, \, x \end{cases}$$

and let there be a $f_x^k(s), k \ldots 0$(, especially for $k = 0$

$$f_x^0(s) = \begin{cases} 1, & \text{for } s > x \\ 0, & \text{for } s \,,, \, x \end{cases}$$

The function $f_x^k(.)$ consists of two polynomials of degree " k:

$$\begin{bmatrix} P_0(s) = 0, & \text{for } s \,,, \, x \\ P_1(s) = (s-x)^k, & \text{for } s > x \end{bmatrix}$$

Note that the function f_x^k depends on the real parameter x(and $f_x^k(s)$ is defined as a function of x(that is continuous on the right for each fixed s(. For $k \ldots 1$, $f_x^k(s)$ is $(k-1)$ times continuously differentiable, i.e. of class $C^{k-1}(R)$ with respect to s(and x(.

The divided difference $f[s_i, s_{i+1}, \ldots, s_{i+k}]$ of the real function $f(s)$, is well defined for each segment $s_i, , s_{i+1}, , \ldots, , s_{i+k}$ of real numbers, even if s_j are not different from each other. The only condition is that f be $(d_j - 1)$ times differentiable for $s = s_j, (j = i, i+1, \ldots, i+k)$, if s_j occurs d_j times in the subsegment $s_i, , s_{i+1}, , \ldots, , s_j$ ending in s_j.

According to,

$$f[s_i, \ldots, s_{i+k}] = \frac{f^{(k)}(s_i)}{k!}, \quad \text{for } s_i = s_{i+k} \tag{1}$$

$$f[s_i, s_{i+1}, \ldots, s_{i+k}] = \frac{f[s_{i+1}, \ldots, s_{i+k}] - f[s_i, \ldots, s_{i+k-1}]}{s_{i+k} - s_i}, \quad \text{for } s \neq s_{i+k} \tag{2}$$

Let $k \ldots 1$ be an integer and $s = \{s_i\}_{i \in Z}$ any infinite distribution of real numbers s_i, where

$$\inf s_i = -\infty, \quad \sup s_i = +\infty \quad \text{and} \quad s_i \,,, \, s_{i+k}, \ (\forall i \in \mathrm{Z})$$

the i-th B-spline of the order k associated with s is defined as a function of x:

$$\begin{aligned} B_{i,k,s}(x) &= (s_{i+k} - s_i) f_x^{k-1}[s_i, s_{i+1}, \ldots, s_{i+k}] \\ &= f_x^{k-1}[s_{i+1}, \ldots, s_{i+k}] - f_x^{k-1}[s_i, s_{i+1}, \ldots, s_{i+k-1}] \end{aligned} \tag{3}$$

Sometimes the text is written shorter only as B_i or $B_{i,k}$.

$B_{i,k,s}(x)$ is well defined function for each $x \neq s_i, s_i, , s_{i+1}, , \ldots, , s_{i+k}$ and according to (3) is a linear combination of functions $(s_i - x)_+^{k-d_i} | s = s_j, \ i, , j, , i+k$ if the value s_i occurs d_j times within the subsegment $s_i, , s_{i+1}, , \ldots, , s_j$.

In the following, we will examine what a B-spline of some order-i looks like.

Let be a given continuous distribution in nodes $s = (s_i)$. B-spline of order 1 in given nodes is a characteristic function of this distribution, i.e., the function

$$B_{i,1}(s) = \chi_i(s) = \begin{cases} 1, & \text{for } s_i \le s < s_{i+1} \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

All of these selected functions are continuous on the right. B-spline of order 1 must give in the sum a one, i.e.

$$\sum_i B_{i,1}(s) = 1, \qquad \text{for every } s.$$

In particular,

$$s_i = s_{i+1} \Rightarrow \quad B_{i,1} = \chi_i = 0.$$

Higher order B-spline can be obtained, using first-order B-spline, by a recurrent relation:

$$B_{i,k} = \rho_{i,k} B_{i,k-1} + (1-\rho_{i+1,k}) B_{i+1,k-1} \tag{5}$$

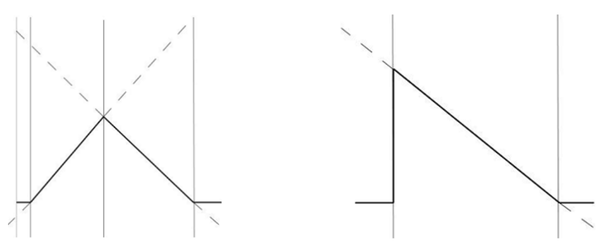
where is

$$\rho_{i,k}(s) = \begin{cases} \dfrac{s - s_i}{s_{i+k-1} - s_i} & \text{for } s_i \neq s_{i+k-1} \\ 0, & \text{otherwise} \end{cases}$$

Therefore, B-spline of order 2 is given by:

$$B_{i,2} = \rho_{i,2}\chi_i + (1-\rho_{i+1,2})\chi_{i+1} \tag{6}$$

and consists, in general, of two nontrivial linear parts that continuously merge to form piecewise linear functions that disappear outside the $[s_i, s_{i+2})$ interval. Therefore, $B_{i,2}$ is called linear B-spline.



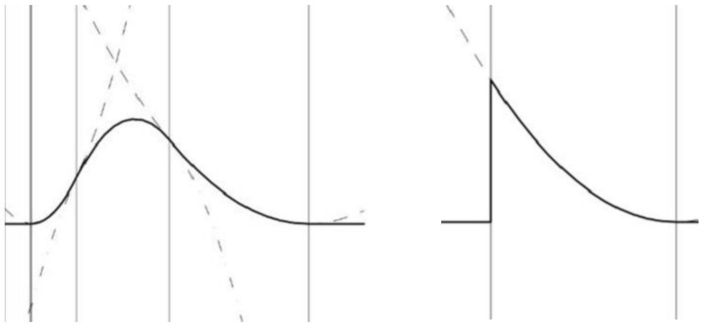
a) b)

Fig 1. B-spline of order 1 (Linear B-spline)
a) single nodes, b) double nodes

Furthermore, we compute B-spline of order 3:

$$\begin{aligned} B_{i,3} &= \rho_{i,3} B_{i,2} + \left(1-\rho_{i+1,3}\right) B_{i+1,2} \\ &= \rho_{i,3}\rho_{i,2}\chi_i + \left[\rho_{i,3}\left(1-\rho_{i+1,2}\right) + \left(1-\rho_{i+1,3}\right)\right]\chi_{i+1} + \left(1-\rho_{i+1,3}\right)\left(1-\rho_{i+2,2}\right)\chi_{i+2} \end{aligned} \tag{7}$$

In general, a B-spline of order 3 consists of 3 nontrivial square parts, and, according to Fig . 2 we see that they merge smoothly at the nodes and form piecewise quadratic functions of class C^1 that disappear outside the $[s_i, s_{i+2})$ interval. If e.g. $s_i = s_{i+1} = s_{i+2})$ (i.e. $X_i = X_{i+1} = 0$), then consists of only one non-trivial part, which is continuous in triple node s_i, but is still class C^1 in single node s_{i+3}, as shown in Fig . 2b).



a) b)

Fig 2. B-spline of order 2 (Square B-spline)
a) single nodes, b) triple nodes

After the (k-1) step, the $B_{i,k}$ is of the form:

$$B_{i,k} = \sum_{j=1}^{i+k-1} \beta_{j,k} \chi_j \quad (8)$$

where each $\beta_{j,k}$ is a polynomial of degree $< k$, because it's the sum of the products of (k-1) linear polynomials. Thus, a B-spline of order k consists of piecewise polynomials of order $< k$ (In fact, all $\beta_{j,k}$ are exactly of degree (k-1)).

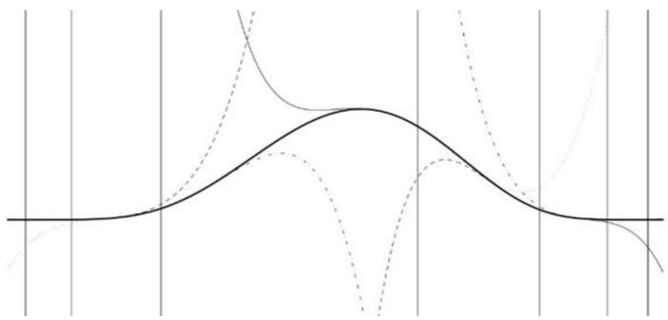


Fig 3. B-spline of order 6 consists of 6 polynomials of degree 5.

From this, we can conclude:

(a) $B_{i,k}$ is a partial polynomial of degree $< k$, that vanishes outside interval $[s_i, s_{i+k})$;

(b) $B_{i,k}$ is a zero-function only in the case $s_i = s_{i+1}$;

(c) $B_{i,k}$ is positive on the open interval (s_i, s_{i+k}), because both $\rho_{i,k}$ and $(1 - \rho_{i+1,k})$ are positive on that interval (Example in Fig . 4).

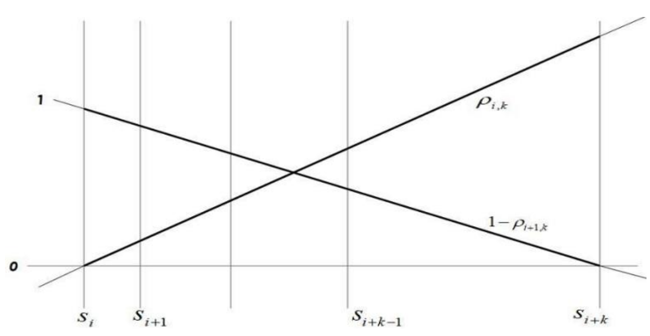


Fig 4. Two ρ functions that are positive on the interval (s_i, s_{i+k})

The basic properties of B-spline are given through the following theorem:

Theorem 1[2]

(a) $B_{i,k,s}(x) = 0$, for $x \notin [s_i, s_{i+k}]$

(b) $B_{i,k,s}(x) > 0$, for $x \in [s_i, s_{i+k}]$

(c) $\sum_i B_{i,k,s}(x) = 1, \forall x R$

The sum in (c), has finite number of summands $B_{i,k,s}(x)$ different from zero.

3. Compute of B-spline

B-spline can be computed using recursion. The recursion is based on the generalization of the Leibniz formula for the derivation of the product of the two functions.

Theorem 2[2]

Let $s_i, , s_{i+1}, , \dots, , s_{i+m}$. Suppose further that the function $f(s) = g(s) \cdot h(s)$ is the product of two functions that differ for some $s = s_j, j = i, i+1 \dots, i+m,$ therefore $g = [s_i, , s_{i+1}, , \dots, , s_{i+m}]$ and $h\ [s_i, , s_{i+1}, , \dots, , s_{i+m}]$ are defined as (1). So it follows:

$$f[s_i, s_{i+1}, \dots, s_{i+m}] = \sum_{j=i}^{i+m} g[s_i, s_{i+1}, \dots, s_j] \cdot h[s_j, s_{j+1}, \dots, s_{i+m}]$$

Using Theorem 2 and relation (03), since $B_{i,k}(x) \equiv B_{i,k,s}(x)$, recursion is obtained:

$$B_{i,k}(x) = \frac{x - s_i}{s_{i+k-1} - s_i} B_{i,k-1}(x) + \frac{s_{i+k} - x}{s_{i+k} - s_{i+1}} B_{i+1,k-1}(x) \qquad (9)$$

Recursion (9) is used to compute all B-spline for a given fixed value x. Thus, for a given value of x, Theorem 1 (a) gives $B_{i,k}(x) = 0$ for each i, k and $x \notin [s_i, s_{i+k}]$ i.e. for $i \dots j+1$. We will present this in Table 1 below, when $B_{i,k}(x)$ disappears in positions where it is zero.

0	0	0	0	$\cdots$
0	0	0	$B_{j-3,4}(x)$	$\cdots$
0	0	$B_{j-2,3}(x)$	$B_{j-2,4}(x)$	$\cdots$
0	$B_{j-1,2}(x)$	$B_{j-1,3}(x)$	$B_{j-1,4}(x)$	$\cdots$
$B_{j,1}(x)$	$B_{j,2}(x)$	$B_{j,3}(x)$	$B_{j,4}(x)$	$\cdots$
0	0	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 1

By definition, for $x \in [s_i, s_{j+1})$, the element $B_{j,1} = B_{j,1}(x) = 1$ determines the first column of Table 1. The remaining columns can be computed consecutively using recursion (19): each $B_{i,k}$ element can be derived using two adjacent elements, $B_{i,k-1}$ and $B_{i+1,k-1}$.

This method is numerically very stable, because only non-negative multiples of non-negative numbers are added together.

Example 1. For $s_i = i, (i = 0, 1, 2, \ldots)$ and $x = 3.5 \in [s_3, s_4]$, the table of values of $B_{i,k}$ is as follows:

Table 2.

$B_{i,k}$	$k=1$	$k=2$	$k=3$	$k=4$
$i=0$	0	0	0	1/48
$i=1$	0	0	1/8	23/48
$i=2$	0	1/2	3/4	23/48
$i=3$	1	1/2	1/8	1/48
$i=4$	0	0	0	0

For example, $B_{1,4}$ was obtained from:

$$B_{1,4} = B_{1,4}(3.5) = \frac{3.5 - s_1}{s_4 - s_1} B_{1,3}(3.5) + \frac{s_5 - 3.5}{s_5 - s_2} B_{2,3}(3.5)$$
$$= \frac{3.5-1}{4-1} \cdot \frac{1}{8} + \frac{5-3.5}{5-2} \cdot \frac{3}{4} = \frac{23}{48}.$$

4. B-spline approximation

The B-spline approximation of the degree $k-1$ (order k) to the arbitrary function $f\colon [0, n] \to R$ is:

$$W_k[f; x] = \sum_i f(\tau_i) B_{i,k}(x) \tag{10}$$

where is

$$\tau_i = \frac{1}{k-1}(s_{i+1} + s_{i+2} +, \ldots, s_{i+k-1}) \tag{10a}$$

Approximation (10) contains k non-zero conditions. B-spline approximation is a local approximation, so it takes into account only the local behavior of the primitive function.

For a given set of values $\{z_0, z_1, \ldots, z_l\}$, we want to find a unique set of functional values $\{f(\tau_0), f(\tau_1), \ldots, f(\tau_l)\}$ such that the B-spline approximation of f interpolates given data, i.e. we want to find $f(\tau_i)$ that satisfies:

$$W_k[f; \tau_i] = z_i, \qquad i = 0, 1, \ldots, l$$

or in matrix form, we obtain the following relation

$$B = F^T Z^T \tag{11}$$

where is

$$B = \begin{bmatrix} B_{0,k}(\tau_0) & B_{1,k}(\tau_0) & \cdots & B_{l,k}(\tau_0) \\ B_{0,k}(\tau_1) & B_{1,k}(\tau_1) & \cdots & B_{l,k}(\tau_1) \\ \vdots & \vdots & & \vdots \\ B_{0,k}(\tau_l) & B_{1,k}(\tau_l) & \cdots & B_{l,k}(\tau_l) \end{bmatrix},$$

$$F = \left(f(\tau_0), f(\tau_1), ..., f(\tau_l) \right),$$

$$Z = \left(z_0, z_1, ..., z_l \right).$$

Matrix B is a Gram matrix, which we know is invertible for different sets of nodes $\{\tau_i\}$. The inverse function formula is then

$$F^T = B^{-1} Z^T \qquad (12)$$

5. Estimate of B-spline

The function F defined on the set $[0, m]$ can be represented in the form[3]:

$$F(x) = \sum_i A_i B_{i,k}(x) \qquad (13)$$

To estimate the B-spline function F in (13) at point $x \in [s_j, s_{j+1})$, without computing the basic functions $B_{i,k}(x)$, it is necessary to compute k numbers

$$B_{i,k}(x), \quad i = j-k+1, ..., j,$$

therefore, $F(x)$ can be written as:

$$F(x) = \sum_{i=j-k+1}^{j} A_i B_{i,k}(x).$$

The function $F(x)$ can also be written using a lower-order B-spline, with certain polynomial coefficients

$$F(x) = \sum_i A_i^{[1]}(x) B_{i,k-1}(x),$$

where is

$$A_i^{[1]}(x) = \frac{x - s_i}{s_{i+k-1} - s_i} A_i + \frac{s_{i+k-1} - x}{s_{i+k-1} - s_i} A_{i-1}.$$

More generally,

$$A_i^{[j]}(x) = \begin{cases} A_i, & j = 0 \\ \dfrac{x - s_i}{s_{i+k-j} - s_i} A_i^{[j-1]}(x) + \dfrac{s_{i+k-j} - x}{s_{i+k-j} - s_i} A_{i-1}^{[j-1]}(x), & j > 0, \end{cases} \qquad (14)$$

we have

$$F(x) = \sum_i A_i^{[j]}(x) B_{i,k-j}(x). \qquad (15)$$

Since $B_{i,1}(x) = 1$ for $x \in [s_i, s_{i+1})$ and zero otherwise, it follows that

$$F(x) = \sum_i A_i^{[k-1]}(x), \qquad s_i ,, \ x < s_{i+1}.$$

Therefore, if $x \in [s_i, s_{i+1})$, then $F(x)$ can be found by $A_{i-k+1}, \dots, A_i$, by creating convex combinations using (14).The algorithm for computing the $A_i^{[j]}(x)$ components is based on (14) and (15), creating the following table

$$
\begin{array}{lllll}
A_{i-k+1}^{[0]}(x) & & & & \\
A_{i-k+2}^{[0]}(x) & A_{i-k+2}^{[1]}(x) & & & \\
\vdots & \vdots & \ddots & & \\
A_{i-1}^{[0]}(x) & A_{i-1}^{[1]}(x) & \cdots & A_{i-1}^{[k-2]}(x) & \\
A_{i}^{[0]}(x) & A_{i}^{[1]}(x) & \cdots & A_{i}^{[k-2]}(x) & A_{i}^{[k-1]}(x)
\end{array}
$$

Table 3

The required $F(x)$ is in the lowest right entry of the table, which is $A_i^{[k-1]}(x)$.

6. Computational examples of B-spline

B-splines are widely used in various applications[5], and one of them is in B-spline curves.
Definition: The B-spline curve of degree $k-1$ (order k) in relation to polygon P is

$$K_m[\mathrm{P}] = \sum_{i=0}^{m} P_i B_{i,k}(x), \quad 0,, \ x_m \tag{16}$$

given by the distribution $s_0, s_2, \dots, s_m$ so that $s_i < s_{i+1}$.
A periodic or closed B-spline curve occurs if the B-spline function is defined by the distribution $s_0, s_1, \dots, s_m$, $s_i < s_{i+1}$, where

$$x_i = x_{(i-k/2) \bmod x_m} \tag{17}$$

The interpretation of the previous algorithm geometrically leads to a constructive method for determining the point of the B-spline curve. Formula (14) in the conditions of the vertices P_i of the polygon P is of the form:

$$P_i^{[j]}(x) = \begin{cases} P_i, & \text{for } j = 0 \\ \mu P_i^{[j-1]}(x) + (1-\mu) P_{i-1}^{[j-1]}(x), & \text{for } j > 0 \end{cases} \tag{18}$$

where is

$$\mu = \frac{x - s_i}{s_{i+k-j} - s_i}.$$

Example 2

Let the closed polygon $P_0 P_1 \dots P_{12}$ $P_0 P_1 \dots P_{12}$ be given, we want to find the points of the B-spline curve of order $k = 4$ corresponding to $x = 7.6, s_i = i$, for $i = 0, \dots, 13$ according to (17) we have:

$$x_i = x_{(i-2) \bmod 13} = (i-2) \bmod 13$$

According to the algorithm (in the Table 3) for $x \in [s_i, s_{i+1})$ so $s_i = 7$ satisfies the condition, i.e. $i = 9$. According to (18) for $j = k-1$ is $P_i^{[k-1]}(x) = P_9^{[3]}$.
Using a recursive algorithm (18), we compute:

$$P_9^{[3]}(7.6) = \mu P_9^{[2]}(7.6) + (1-\mu) P_8^{[2]}(7.6)$$

where $\mu = \dfrac{x - s_9}{s_{10} - s_9} = \dfrac{7.6 - 7.0}{8 - 7} = 0.60$;

$$P_9^{[2]}(7.6) = \mu P_9^{[1]}(7.6) + (1-\mu) P_8^{[1]}(7.6)$$

where $\mu = \dfrac{x - s_9}{s_{11} - s_9} = 0.30$;

$$P_8^{[2]}(7.6) = \mu P_8^{[1]}(7.6) + (1-\mu) P_7^{[1]}(7.6)$$

where $\mu = \dfrac{x - s_8}{s_{10} - s_8} = 0.80$;

$$P_9^{[1]}(7.6) = \mu P_9 + (1-\mu) P_8$$

where $\mu = \dfrac{x - s_9}{s_{12} - s_9} = 0.20$;

$$P_8^{[1]}(7.6) = \mu P_8 + (1-\mu) P_7$$

where $\mu = \dfrac{x - s_8}{s_{11} - s_8} = 0.53$;

$$P_7^{[1]}(7.6) = \mu P_7 + (1-\mu) P_6$$

where $\mu = \dfrac{x - s_7}{s_{10} - s_7} = 0.87$.

In Fig .5 we see the given vertices P_i of the polygon P and the corresponding interpolation of our example.

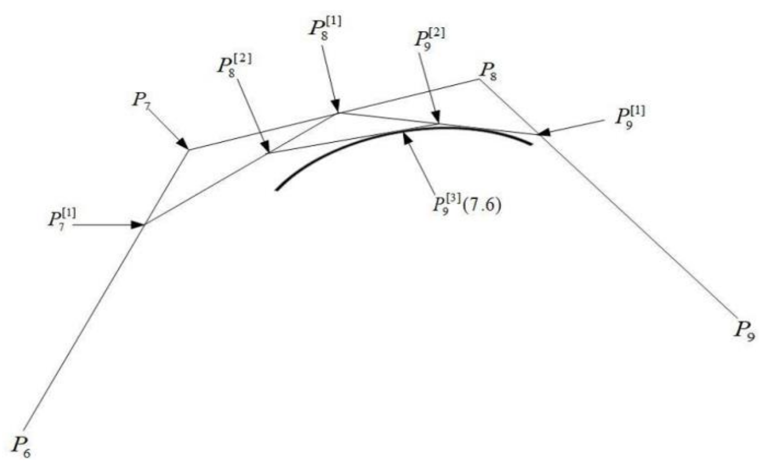


Fig 5.

In addition, if we assume that the polygon P is a piecewise linear function F , where F is defined so that the B-spline curve is a parametric approximation of the B-spline to F, then according to (10) and (10a) F write as:

$$F(\tau_i) = P_i, \quad i = 0,1,...,m$$

where $$\tau_i = \frac{1}{k-1}\left(s_{i+1} + s_{i+2} + \cdots + s_{i+k-1}\right) \qquad (10a)^*$$

In this regard, we have the following example:

Example 3

Let the given open B-spline of order $k = 4$ be determined by the polygon $P_0P_1 \dots P_5$ and

$s_0 = s_1 = s_2 = s_3 = 0 < s_4 = 1 < s_5 = 2 < s_6 = s_7 = s_8 = s_9 = 3$

We will first, using $(10a)^*$, compute the values of τ_i:

$$\tau_0 = \frac{1}{3}\left(s_1 + s_2 + s_3\right) = 0; \qquad \tau_1 = \frac{1}{3}\left(s_2 + s_3 + s_4\right) = \frac{1}{3};$$

$$\tau_2 = \frac{1}{3}\left(s_3 + s_4 + s_5\right) = 1; \qquad \tau_3 = \frac{1}{3}\left(s_4 + s_5 + s_6\right) = 2;$$

$$\tau_4 = \frac{1}{3}\left(s_5 + s_6 + s_7\right) = \frac{8}{3}; \qquad \tau_5 = \frac{1}{3}\left(s_6 + s_7 + s_8\right) = 3$$

The function F then has the value:

$$F(0) = P_0, \quad F(1/3) = P_1, \quad F(1) = P_2, \quad F(2) = P_3, \quad F(8/3) = P_4, \quad F(3) = P_5.$$

7. Conclusion

The aim of this paper is to define B-spline in a way that clarifies its basic properties, and thus to apply it in some specific numerical procedures. Through the divided differences, a B-spline of a certain order was determined, which was easier to compute with a numerically stable recursive algorithm. Then, it was shown that the choice of B-splines is the most suitable for approximation of functions. The numerical example shows the B-spline curve, ie determining the point of the B-spline curve considering the vertices of a given polygon.

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