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CHARACTERIZATION OF ISOLATED POINTS IN T_1 SPACES USING CHAINS

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Abstract

By the standard definition, a point x is an isolated point in a topological space if its corresponding one-element set is open. Here it is characterized the notion of isolated point using chains and open covers of the space. Namely, x is an isolated point in a T_1 topological space if there exists an open covering of the space such that for any point of the space different from x, there is no chain in the covering joining it with the point x. Equivalently, it is provided a characterization of isolated point using the notion of pair of chain separated sets relatively a T_1 space, while when using the notion of pair of weakly chain separated sets relatively a T_1 space it is shown that only the sufficiency of the claim holds.
Using these results, it is provided a characterization of discrete topological spaces and it is reproved that every compact Hausdorff space without isolated points is uncountable.

Keywords: Isolated point, Chain connectedness, Chain separatedness, Open cover.

1. Introduction

Throughout this text by space we mean topological space and by covering we mean open covering.
In [3], using the notion of chain and coverings, the notion of chain connected topological space is introduced. In the same paper it is proved the equivalence of this notion with the notion of connectedness of the space. In [1] and [4] the notion of chain connectedness of a set relatively to the space is introduced.
Let X be a topological space and let U be a covering.
For any two points $x, y \in X$ we say that they are U-chain related if there exists a finite family $U_1, U_2, \dots, U_n$ in U such that $x \in U_1, y \in U_n$ and

$$U_i \cap U_{i+1} \neq \emptyset, \forall i \in \{1, 2, \dots, n-1\}.$$

We denote that with $x \underset{\mathrm{U}}{\sim} y$. This relation is an equivalence relation. For a fixed $x \in X$, the class represented by x will be denoted by $V_{\mathrm{U}}(x)$. In [1] it is proved that the classes of equivalence are clopen sets.
The family $U_1, U_2, \dots, U_n$ is called a chain in U joining x and y. If such a chain does not exist, we say that x and y are not U-chain related and we denote that with $x \underset{\mathrm{U}}{\not\sim} y$.
If $x, y \in X$ are U-chain related for any covering U, then we say that they are chain related and we denote it with $x \sim y$. This relation is also an equivalence relation and its classes of equivalence are called chain components. For a fixed $x \in X$, the class represented by x will be denoted by $V(x)$. In [3] it is proved that the chain components coincide with the quasicomponents of the space. If there is a covering U such that there is no chain in U joining x and y, we say that x and y are not chain related and we denote that with $x \not\sim y$.

Definition 1.1. [1, 4] For a given set $A \subseteq X$ we say that it is chain connected relatively the space X (or shortly: chain connected) if any two points $x, y \in A$ are chain connected in X.

Proposition 1.2. [1] If A is a chain connected set relatively a space X, then each subset of A is chain connected set relatively X.

Proposition 1.3. [4] If $B \subseteq A \subseteq \overline{B}$ and B is chain connected set relatively a space X, then A is also chain connected set relatively X.

The following theorem is a generalization of Remark 2.2 in [1] and Theorem 12 in [4].

Theorem 1.4. If $B \subseteq \overline{A}$ and A is chain connected set relatively a space X, then B is also chain connected set relatively X.

Proof. A direct consequence of Proposition 1.1 and Proposition 1.2.

The following results are proved in [1] and/or [4].

Theorem 1.5. A connected set is chain connected relatively the space. The converse may not be true.

Theorem 1.6. If a set is chain connected set relatively a space X, then it is a chain connected set relatively Y for every superspace Y of the set such that $Y \subseteq X$.

Theorem 1.7. A set A is connected if and only if it is chain connected relatively A.

Theorem 1.8. If A is a chain connected set relatively X and $f : X \to Y$ a continuous function, then $f(A)$ is chain connected relatively Y.

Theorem 1.9. [5] If A_i are chain connected sets relatively X_i, $\forall i \in I$, then $\prod_{i \in I} \overline{A}$ is a chain connected set relatively $\prod_{i \in I} X$ equipped with the product topology.

In [1] the notion of chain separated sets relatively the space is introduced.

Definition 1.10. The nonempty sets $A, B \subseteq X$ are chain separated relatively X if there exists a covering U of X such that for every point $x \in A$ and every $y \in B$, there is no chain in U that connects x and y.

Theorem 1.11. [1] A set A is chain connected relatively X if and only if A cannot be represented as a union of two chain separated sets relatively X.

For a pair of sets in a topological space, the notion of weakly chain separateness is introduced in [6].

Definition 1.12. The nonempty sets $A, B \subseteq X$ are weakly chain separated relatively X if for every point $x \in A$ and every $y \in B$, there exists a covering $\mathrm{U} = \mathrm{U}(x, y)$ of X such that there is no chain in U that connects x and y.

As name suggests, the notion of weakly chain separateness is a weaker notion than that of chain separateness.

2. Characterizations of isolated point by using the notion of a chain

The point $x \in X$ is an isolated point of the space X if the singleton $\{x\}$ is open in X.
Let X be a T_1 space.

Theorem 2.1. The point $x \in X$ is an isolated point of the space X if and only if there exists a covering $\mathcal{U}$ of X such that $x \underset{\mathcal{U}}{\not\sim} y$, for any $y \in X \setminus \{x\}$.

Proof. Let $x \in X$ be an isolated point of the space X. Then $\{x\}$ is open in X. Since X is a T_1 space, $\{x\}$ is closed in X. Then $\mathcal{U} = \{\{x\}, X \setminus \{x\}\}$ is a covering such that $x \underset{\mathcal{U}}{\not\sim} y$, for any $y \in X \setminus \{x\}$.

Conversely, let $\mathcal{U}$ be a covering of the space X such that $x \underset{\mathcal{U}}{\not\sim} y$, for any $y \in X \setminus \{x\}$. Then $V_{\mathcal{U}}(x) = \{x\}$ is clopen hence x is an isolated point of X.

Remark 2.2. Notice that for the necessity condition of the theorem, X does not have to be a T_1 space. However, the following example tells us that T_1 is crucial for the sufficient condition of the theorem.

Example 2.3. Let $X = \{a,b,c\}$ be a space equipped with the topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$. Then X is not a T_1 space (there is no open set that contains c but not a), a is an isolated point of X, but for every possible covering $\mathcal{U}$ of X we have $a \underset{\mathcal{U}}{\sim} c$.

Corollary 2.4. If x is an isolated point of the T_1 space X, then the chain component containing x is a singleton, i.e. $V(x) = \{x\}$.

Let A be a subset of the T_1 space X. The point $x \in A$ is an isolated point of the set A if there exists a neighborhood U of x such that $(U \setminus \{x\}) \cap A = \emptyset$.

Corollary 2.5. The point $x \in A$ is an isolated point of the set A in the space X if and only if there exists a covering $\mathcal{U}$ of X such that $x \underset{\mathcal{U}_A}{\not\sim} y$, for any $y \in A \setminus \{x\}$, where $\mathcal{U}_A = \{U \cap A \mid U \in \mathcal{U}\}$.

One may wonder if the corollary will hold if you replace $\mathcal{U}_A$ with a covering of A consisting of open sets in X. This is not true in a general case. As a counterexample, any rational number is an isolated point of the set $\mathbb{Q}$ in the Euclidean space $\mathbb{R}$. However, for an open covering $\mathcal{U}_A$ of $\mathbb{Q}$ consisting of open sets of $\mathbb{R}$, if $x \in U \in \mathcal{U}_A$, then there exists $y \in \mathbb{Q} \setminus \{x\}$ such that $y \in U$.

3. Characterizations of isolated point by using the notion of chain separated and separated sets.

Next, we characterize isolated points using the notion of pair of chain separated sets relatively the space.
Let X be a T_1 space.

Theorem 3.1. The point $x \in X$ is an isolated point of the space X if and only if $\{x\}$ and $X \setminus \{x\}$ are chain separated relatively X.

Proof. ($\Rightarrow$) Let x be an isolated point of the space X. It follows that $\{x\}$ is an open set. Since X is a T_1 space, $\{x\}$ is also a closed set. Therefore, for the covering $\mathrm{U} = \{\{x\}, X \setminus \{x\}\}$ there is no chain in U that connects x and y, for every $y \in X \setminus \{x\}$. Hence $\{x\}$ and $X \setminus \{x\}$ are chain separated sets relatively X.
($\Leftarrow$) Let $\{x\}$ and $X \setminus \{x\}$ be chain separated sets relatively X. Then there exists a covering U of X such that $x \underset{\mathrm{U}}{\not\sim} y$ for every $y \in X \setminus \{x\}$. Then $V_{\mathrm{U}}(x) = \{x\}$ is clopen in X, thus x is an isolated point of X. □
Since chain separated sets in their union are separated sets [1], the next corollary which is a criteria for isolated point by using the notion of separated sets, is proven.

Corollary 3.2. The point $x \in X$ is an isolated point of the space X if and only if $\{x\}$ and $X \setminus \{x\}$ are separated.
However, since weak chain separateness is a weaker notion than chain separateness of pairs of sets relatively the space, this leads to the following statement.
Let X be a T_1 space.

Corollary 3.3. If the point $x \in X$ is an isolated point of the space X, then $\{x\}$ and $X \setminus \{x\}$ are weakly chain separated sets relatively X.
The next example shows that the converse statement of the corollary does not hold in general.

Example 3.4. The topological space:

$$X = A \cup B \text{, where } A = \{0\} \text{ and } B = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\},$$

is a T_1 space. The sets $A = \{0\}$ and $B = X \setminus \{0\}$ are weakly chain separated relatively X, but they are not chain separated relatively X . Hence by Theorem 3.1, 0 is not an isolated point of X.
Namely, if $b \in B$. Then $b = \frac{1}{n_0}$ for some $n_0 \in \mathbb{N}$, and for the covering:

$$\mathrm{U} = \left\{ \left[0, \frac{1}{n_0}\right) \cap X, \left\{\frac{1}{n_0}\right\}, \left\{\frac{1}{n_0 - 1}\right\}, \ldots, \left\{\frac{1}{2}\right\}, \{1\} \right\},$$

there is no chain in U that connects 0 and b. It follows that A and B are weakly chain separated in X.
On the other hand, since the topology on X is relative to $\mathbb{R}$, every element of an arbitrary covering of X that contains the point 0, also contains a point from the set B. It follows that A and B are not chain separated in X. ■

4. Characterization of the discrete space

The topological space is the discrete if every point is an isolated point of the space. In [6] the discrete space is characterized using the notion of pair of chain separated sets.

Theorem 4.1. [6] The space X is the discrete if and only if any two disjoint nonempty subsets of X are chain separated relatively X.
The following theorem improves the conditions of Theorem 4.1.

Theorem 4.2. The space X is the discrete if and only if for every $x \in X$, the subsets $\{x\}$ and $X \setminus \{x\}$ are chain separated relatively X.

Proof. $(\Rightarrow)$ Let X be the discrete space. Since any subset of the discrete space is an open set, for an arbitrary $x \in X$, there is no chain in $\mathrm{U} = \{\{x\}, X \setminus \{x\}\}$ that connects x and y, for every $y \in X \setminus \{x\}$. It follows that $\{x\}$ and $X \setminus \{x\}$ are chain separated relatively X.

$(\Leftarrow)$ If for any point $x \in X$, the subsets $\{x\}$ and $X \setminus \{x\}$ are chain separated relatively X, then $V_{\mathrm{U}}(x) = \{x\}$ is clopen i.e. $\{x\}$ is an isolated points of X. Hence X is the discrete space.□

As a corollary, the discrete space is characterized by using the notion of separated sets.

Corollary 4.3. The space X is the discrete if and only if any for every $x \in X$, the subsets $\{x\}$ and $X \setminus \{x\}$ are separated.

Remark 4.4. Notice that the discrete space is a T_1 space. Therefore, that condition is not required in the theorems in which the discrete space is characterized by the notion of chain.

5. Isolated points on Hausdorff spaces

The following results can be found in [2]. Here we prove the same using the notion of chain connectedness.

Theorem 5.1. Let X be a Hausdorff space without an isolated point. Let U be an open nonempty set of X and let $x \in X$. Then there exists a nonempty open set V contained in U such that $x \notin \bar{V}$.

Proof. We can always choose $y \in U$ such that $y \neq x$: if $x \notin U$, there exists $y \in U$ since U is nonempty; if $x \in U$ suppose that there is no $y \in U$ such that $y \neq x$, i.e. $U = \{x\}$, but since the space is Hausdorff, $\{x\}$ is closed, hence $\mathrm{U} = \{U, X \setminus U\}$ is a cover of X such that $x \underset{\mathrm{U}}{\nsim} y$, for any $y \in X \setminus \{x\}$, hence x is an isolated point (Theorem 2.1) which is a contradiction.

Let W_x and W_y be disjoint neighborhoods of x and y, respectively. Then $W_y \cap U$ is a nonempty open subset of U whose closure does not contain x. □

Theorem 5.2. [2] Let X be a nonempty compact Hausdorff space without any isolated point. Then no function $f : \mathbb{N} \to X$ can be surjective.

Corollary 5.3. Let X be a compact Hausdorff space without an isolated point. Then X is uncountable.

6. Conclusion

The standard definition of isolated point is given using neighbourhoods. Here we used coverings of a T_1 space to characterize isolated points. Some equivalent claims were provided using notions related to chain connectedness. This new perspective can be used on solving problems where isolated points are involved. As an illustration, the discrete space was characterized and a theorem for compact Hausdorff spaces with no isolation points was reproved.

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