UDC: 515.124
Professional paper

METRIZABILITY OF TOPOLOGICAL SPACES

Ferzije LLESHI POLLOZHANI[1*], Krutan RASIMI[1], Flamure SADIKI[1], Bedrije BEXHETI[1]

[1]Department of Natural Sciences and Mathematics, University of Tetova, NMK
[*]Corresponding author e-mail: ferzije.lleshi@unite.edu.mk

Abstract

In this paper, we are going to analyze a wide class of topological spaces - the so-called metric spaces. These are sets in which is defined the notion of distance between any two points. It is well known that the distance function or metric defined on a metric space X induces a topology on that space X.
In the second part of the paper, we are going to study well-known characterizations of the class of topological spaces, the topology of which is determined with the help of metrics - the so-called metrizable spaces.
The question here is when is a topological space metrizable?
The answer of this question is the main result of this research, which are some important criteria, who are necessary and sufficient, that topological spaces must possess in order to be metrizable.

Keywords: Topological spaces, Metric spaces, Metrizable spaces, T3-space, Hilbert space.

Introduction

1. Metric and Topological Spaces

1.1. Metric Spaces:

Definition 1.1.1 Let X be any non-empty set. A function $d: X \times X \to \mathbb{R}$ is said to be a metric or a distance on X if the following conditions are satisfied:

(M1) $d(x, y) \geq 0$ for all $x, y \in X$;
(M2) $d(x, y) = 0 \Leftrightarrow x = y$;
(M3) $d(x, y) = d(y, x)$ for all $x, y \in X$ (Symmetry);
(M4) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$ (Triangle inequality).

The pair (X, d) is called a **metric space** and the elements of X are called points.

Example 1.1.1

- Let $X = \mathbb{R}^n$. One can define on X the following distances:

for any two points $x = (x_1, x_2, \dots, x_n)$ *and* $y = (y_1, y_2, \dots, y_3)$ of X;

$$d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i| , \ d_2(x, y) = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2} \text{ and}$$

$$d_\infty(x, y) = sup\{|x_i - y_i| : i = 1,2, \dots, n\}.$$

These three distances are equivalent, i.e. there are real constants k_1, k_2, k_3 such that:

$$d_\infty(x, y) \le k_1 d_2(x, y) \le k_2 d_1(x, y) \le k_3 d_\infty(x, y).$$

- Let X be any arbitrary non-empty set and let us define for $x, y \in X$

$$d(x,y) = \begin{cases} 1 & \text{for } x \neq y, \\ 0 & \text{for } x = y, \end{cases}$$

One can see that d is a metric in X, called *discrete metric* and *(X, d)* the *discrete metric space*.

Definition 1.1.2 Let (X, d) be metric space. Then

1. The distance from a point $x_0 \in X$ to a non-empty set $A \subseteq X$ is defined by:

$d(x_o, A) = \inf\{d(x_0, x) : x \in A\}$

2. The distance between non-empty subsets $A, B \subseteq X$ is defined by:

$d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$

3. The diameter of the non-empty set is defined by:

$diam(A) = \sup\{d(x, y) : x, y \in A\}$.

1.2. Topological Spaces:

Definition 1.2.1 A topological space is a pair $(X, \Im)$ consisting of a non-empty set X and a family $\Im$ of subsets of X satisfying the following conditions:

[T1] $\varnothing \in \Im$ and $X \in \Im$.

[T2] The union of any family of sets of $\Im$ is again in $\Im$.

[T3] The intersection of any two (and hence any finite number of) sets of $\Im$ is again in $\Im$.

The family $\Im$ is called a topology for *X*, and its members are called open sets of *X*. Hence the statements "$G \in \Im$" and "*G* is open in $\Im$" mean the same thing. Elements of *X* are called points.

Example 1.2.1

- Let X be any non-empty space and $\Im = \{\emptyset, X\}$. Clearly, the axioms for a topology (T1),(T2),(T3) given above hold. $\Im = \{\emptyset, X\}$ is a topology in X called the *indiscrete topology*.
- Let X be any non-empty set and let $\Im = P(E)$. Then $\Im$ is a topology in X called the *discrete topology.*
- Let X = {a, b, c} and $\Im = \{\emptyset, \{b\}, X\}$. It is easily verified that $\Im$ is a topology in *X*.
- The real line. Let $X = R$. Define the family $\Im$ as follows:

$U \in \Im \Leftrightarrow$ for any $x \in U$, $\exists \delta x > 0$ such that $u \in U$ if $|x - u| < \delta x$. $\Im$ is a topology called *usual topology* in *R*.

- Let $X = \{a, b\}$ and $\Im = \{\varnothing, X, \{a\}\}$. It is easily verified that $\Im$ is a topology in *X*.

$\Im$ is a topology called *Sierpinski topology* and $(X, \Im)$ is called the *Sierpinski space*.

Definition 1.2.2 Base of a topological space $(X,\Im)$.

A family $B \in \Im$ is called a *base* of the topological space $(X,\Im)$ if every non-empty open subset of X can be written as the union of the members of B i.e.

B *base* of $(X,\Im)$ if for every open set $G \in \Im \setminus \{\emptyset\}$, $G = \bigcup_{i\in I} B_i$, where $B_i \in \mathcal{B}$ $\forall\ i \in I$.

Example 1.2.2

The family of sets with one element $B = \{\{x\} : x \in X\}$ is the base of discrete space (X,D), for $B \subseteq D$, and $\forall\ G \in D \Rightarrow\ G = \bigcup_{x\in G} \{x\}$.

Definition 1.2.3 Sub-base of a topological space $(X,\Im)$.

Let $(X,\Im)$ be a topological space. A sub-collection S of $\Im$ is said to be a *subbase* for $\Im$ if the set B = {*B* |*B* is the intersection of finitely many members of S } is a base for $\Im$.

Example 1.2.3

In *R*, the collection of all open intervals is a base for the usual topology. For a sub-base, one can take the open intervals of the form (−∞, a) and (b, ∞), since any open interval is either one of these or else the intersection of the two of them, i.e. if a < b, (a, b) = (−∞, b) ∩ (a, ∞); and any open set in *R* is a union of open intervals.

Proposition 1.2.1

A subset G of a topological space *X* is *open* if and only if it is a neighborhood of all its points. That is for all x ∈ *G*, we can always find an open set U_x such that x ∈ $U_x \subset G$.

Definition 1.2.4 A subset *F* of a topological space $(X,\Im)$ is said to be *closed* if its complement $X \setminus F$ is open.

Proposition 1.2.2 Let X be any set and suppose that *F* is a family of closed subsets of *X*. The *F* has the following properties,

(F1) ∅ ∈ *F* and X ∈ *F*,
(F2) the union of any two (and hence any finite number of) sets of *F* is again in *F*,
(F3) The intersection of any family of sets of *F* is again in *F*.

Definition1.2.5 Let $(X,\Im)$ be topological space and $Y \in X$. Topology $\Im_y = \{H = G \bigcap Y : G \in \Im\}$ in Y is called *relative topology* in *Y* or induced topology. Topological space $(Y,\Im_Y)$ is called the *subspace* of topological space $(X,\Im)$.

2. Metrizable space

2.1. Topology determined by metric:

Let (X,d) be metric space. One can define the open ball and the closed ball with center x and radius r > 0 respectively by:

$$B_d(x,r) = \{y \in X: d(x,y) < r\}\text{-}\textit{open ball}$$

$$B_d[x,r] = \{y \in X: d(x,y) \le r\}\text{-}\textit{closed ball.}$$

The set $S_d(x,r)=\{y \in X : d(x,y)=r\}$ is called the *sphere* in space (X,d) with center x and radius r > 0.

Example 2.1.1 Let (X, d) be the discrete metric space, with

$$d(x,y) = \begin{cases} r_0 & \text{for } x \neq y, \\ 0 & \text{for } x = y, \end{cases}$$

Then, $B(x,r) = \{x\}$ for any $r < r_0$ and $\mathrm{B}(x,r) = X$ for any $r \ge r_0$.

Theorem 2.1.1 The family $\mathrm{B}=\{\mathrm{B}(x,r): x\in X,\ r>0\}$ of all open balls of metric space (X,d) is the base of a single topology in X.

Definition 2.1.1 Let (X,d) be metric space. The (single) topology in X, which is base is the family $\mathrm{B}=\{\mathrm{B}(x,r): x\in X,\ r>0\}$ of all open balls, is called *metric topology* in X or the *topology induced by the metric d*, such topology is denoted by $\Im(d)$.
The space $(\mathrm{X},\Im(d))$ is called *metric topological space*.

Example 2.1.2

- The topology $\Im(d)$ defined by the *discrete metric* in X is the *discrete topology* in X because $\forall$ $x\in X$, $\mathrm{B}(x,r)=\{x\}$ for any $r<r_0$, is open set.
- Since the open balls in $(\mathbb{R}^2, d_2)$ are open circles in $\mathbb{R}^2$ and the family of such circles forms is the base of the usual topology U in $\mathbb{R}^2$, we conclude that $\Im(d_2)=U$. Similarly it is seen that the Euclidean metric d_2 in $\mathbb{R}^n$ defines the usual topology in $\mathbb{R}^n$, $n\ge 1$.

Until, every metric space is a topological space, then the inverse question is: ***let be*** $(X,\Im)$ ***topological space, does there exist a metric*** d ***in*** X ***which it defines the topology*** $\Im$***, i.e. such that*** $\Im(d)=\Im$***?*** In the general case the answer to this question is ***no***. let's give an example:

Example 2.1.3

There is no metric in the set $X = \{0,1\}$ which determines the topology $\Im = \{X, \emptyset, \{0\}\}$ of Sierpinski space. Really, let's be $d : X \times X \to \mathbb{R}$ any metric in X. Since $X \times X = \{(0,0),(0,1),(1,0),(1,1)\}$ and since $d(x,x) = 0$ and $d(x,y) = d(y,x)$ we conclude that $d(0,0) = d(1,1) = 0$ and $d(0,1) = d(1,0) = r_0$.

So,

$$d(x,y) = \begin{cases} r_o \text{ për } x \neq y \\ 0 \text{ për } x = y \end{cases} \quad x, y \in X = \{0,1\}.$$

Thus we showed that the only metric in the two-element set X is the discrete metric, hence $\Im(d)$ is the discrete topology in X which it is different from topology $\Im$ of Serpinski space.

Definition 2.1.2 Topological space is called *metrizable* $(X, \Im)$ if there exists a metric d in X which it defines the topology $\Im$, i.e. such that $\Im(d) = \Im$.

Consequently from the example above (see example 2.1.2) we conclude that any discrete space X is *metrizable space* because the discrete metric in X defines the discrete topology in X.

Also, the Euclidean space $(\mathbb{R}^n, U)$ *is metrizable space* because the Euclidean metric d_2 in $\mathbb{R}^n$ defines the usual topology U in $\mathbb{R}^n$.

On the other hand, the Sierpinski space is not metrizable (see example 2.1.3).

Theorem 2.1.2 For any subset A of metrizable space X applies:

(1) $\text{Int}A = \{x \in X : d(x, X \setminus A) > 0\}$;

(2) $\overline{A} = \{x \in X : d(x, A) = 0\}$

where d is the metric that defines the topology of space X.

2.2. Some properties of metrizable spaces:

-The continuous bijective function of topological spaces $f : X \to Y$ whose inverse $f^{-1} : Y \to X$ is continuous is called ***homeomorphism*** or ***topological function***. We say that topological space X is homeomorphic or topologically equivalent to topological space Y and we denote it $X \approx Y$, if there exists a homeomorphism $f : X \to Y$.

- The property V of topological space X is called topological property or topological invariant if that property also has any homeomorphic space with space X.

-The property V of a topological space X is called an inherited property if every subspace of the space X also has that property.

Theorem 2.2.1 Metrizability is a *topological property*. If $h: X \to Y$ is the homeomorphism of the topological space $(X, \Im)$ over the metrizable space $(Y, \Im(d))$, then there exists metrics d' in X such that $\Im = \Im(d')$.

Theorem 2.2.2 Metrizability is an *inherited property*. If $(X, \Im(d))$ is metric space and $(Y, \Im(d)_Y)$ it is subspace, then the relative topology $\Im(d)_Y$ in Y is defined by the metric $d_Y = d|_{Y\times Y}: Y \times Y \to \mathbb{R}$, i.e $\Im(d)_Y = \Im(d_Y)$.

Theorem 2.2.3 Every metrizable space is normal space and T_1 (i.e T_4-space).

Theorem 2.2.4 Every metrizable space is:

(1) T_3 space (i.e., regular and T_1-space)
(2) Hausdorf space

Theorem 2.2.5 Every metrizable space X is the first countable space.

Definition 2.2.1 A space *X* is said to be second countable if it has a countable base, i.e. there is a countable collection of open sets such that any open set can be expressed as a union of sets from this collection.
The following example shows that metrizable space is not necessarily the second countable space.

Example 2.2.1 The uncountable discrete space is metrizable space but, it is not the second countable space.

3. Main Results

3.1. Some theorems about the metrizability of topological spaces: The main question is; when is a topological space metrizable? A theorem which answers to that question is called a Metrization Theorem. One of the most important Metrization Theorems is the Urysohn's Metrization Theorem[4] which gives the criteria which suffice for metrizability, while further the class of metrizable spaces is expanded and given by the main theorem of metrization, Smirnov's theorem.[5]

Definition 3.1.1 Let be $I = [0,1]$ the unite segment of the real line $\mathbb{R}$ and let $I^M = \prod_{m\in M} I_m$ be the topological product of the segments $I_m = I$ for each $m \in M$. I^M is called the weight cube $k(M)$, where $k(M)$ is the cardinal number of the set M. In particular, the cube $I^{\mathbb{N}}$, where $\mathbb{N}$ is the set of natural numbers, is called the *Hilbert cube*. So, $I^{\mathbb{N}} = \prod_{n\in\mathbb{N}} I_n$, $I_n = I = [0,1]$ for any $n \in \mathbb{N}$.

Corollary 3.1.1 Hilbert cube defined by $I^{\mathbb{N}} = \prod_{n\in\mathbb{N}} I_n$, $I = [0,1]$ for any $n \in \mathbb{N}$,is metrizable space**.**

Theorem 3.1.1 (*Urysohn*) The necessary and sufficient condition for the topological space to be metrzable is for it to be a T_3 space that satisfies the second axiom of countability.

Proof. Since the Hilbert cube $I^{\mathbb{N}}$ is metrizable space and since metrizability is an inherited property and topological property(theorems 3.2.1 and 3.2.2) it follows that space X is homeomorphic to a subspace of the

Hilbert cube or X can be placed topologically in the Hilbert cube, so from the theorem which says*: X is* T_3 *space, which satisfies the second axiom of countability, then and only then, when X can be placed topologically in the Hilbert cube,* than X is a T_3 space that completes the second axiom of countability. ■

Theorem 3.1.2 For every space X that is T_1 space these conditions are equivalent:

(1) X is the regular space that completes the second axiom of countability,
(2) X is homeomorphic to a subspace of the Hilbert cube $I^{\square}$,
(3) X is separable metric space.

The Urysohn's Metrization Theorem gives only sufficient conditions for the metrizability of topological spaces, but does not give necessary conditions. The next metrization theorem, proved independently by Nagata and Smirnov gives a complete answer to the metrization theorem.

Definition 3.1.2

-The family G of subsets of topological space X is called ***discrete*** if for each $x \in X$ there exists the neighborhood U_x of the point x at X , which has an nonempty intersection with at most one member of family G .

-The family G is called $\sigma -$***discrete*** if it is equal to the union of a countable number of discrete families.

- The family G is called ***locally finit*e** if for each $x \in X$ there exists the neighborhood U_x of the point x at X , which has an nonempty intersection with at most a finite number members of family G .

- The family G is called $\sigma -$***locally finite*** if it is equal to the union of a countable number of locally finite families.

Theorem 3.1.3 $(Bing - Nagata - Smirnov)$

Let be X regular space and T_1 space . Then, the following propositions are equivalent:

(1) X is metrizable space;
(2) X has a $\sigma -$discrete base;
(3) X has the $\sigma -$locally finite base.

References

[1]. Helen F. Cullen. 1967. Introduction to General Topology. *D.C. Heath and Company*, Boston.
[2]. Q. Haxhibeqiri.1989. Topologjia, *ETMM*, Prishtinë.
[3]. Sze-Tsen Hu. 1969. Introduction to General Topology. *Holden-day Series in Mathematics*, Los Angeles.
[4]. Zbigniew Fiedorowicz. 2002. Lectures notes on Urysohn's Metrization Theorem. *Department of Mathematics, Ohio State University, Columbus*, OH 43210, USA.
[5]. Stephen Willard. 2004. General Topology. *Dover Publications*, Inc. New-York.
[6]. Narcisse Roland Loufouma Makala.2007. Metrizability of Topological Spaces, *African Institute for Mathematical Sciences.*
[7]. I.M. Singer and J.A. Thorpe. 1967. Lecture Notes on Elementary Topology and Geometry. *Springer-Verlag*, New-York.
[8]. John L. Kelley.1963. General Topology. *D.Van Nostrand Company*, Inc. Priceton, New Jersey.
[9]. W.A. Sutherland.2003. Introduction to Metric and Topological Spaces. *Oxford Science Publications*, Oxford.