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SOME PROPERTIES OF SEMIGROUP PRESENTATIONS

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Abstract

Semigroup presentations have been studied for a long period. Since 1936 J. A. and H. S. M. have created Todd – Coxeter algorithm for solving enumeration problems of presentations. Here we present some semigroup presentations in terms of generators and defining relations. The aim of this paper is to investigate some properties about the relationship among abstract generators defined on semigroup, using computational methods and to study relationship between some semigroup and group presentation.

Keyword: Semigroup presentations, Generators, Relations, Groups

1. Introduction

Definition 1.1. [3, p.12] A set S together with a binary operation, usually called multiplication is a grupoid. A grupoid S satisfying the associative law is a semigroup.

Definition 1.2. [3, p.12] An element e of S is a left(right) identity of S if $es = s$(se=s) or all $s \in S$. Further, e is two-sided identity of S if it is both a left and a right identity of S. A semigroup with an identity is a monoid.

Definition 1.3.[3, p.13] If A is a nonempty subset of S, then the intersection of all subsemigroups of S containing A is the subsemigroup T of S generated by. If $T = S$, S is generated by A and A is a set of generators for S.

Proposition 1.4.[2, pg.22]. Let E be an equivalence on a semigroup S. The equivalence E is a congruence on S if and only if (xs, yt) ∈ E $(xs, yt) \in E$ for all pairs $(x, y), (s, t) \in E$.

Definition 1.5. [2, pg.22] Let S be a semigroup and let ρ be a congruence on S. The quotient semigroup S/ρ is the semigroup whose elements are the congruence classes of ρ and whose operation $*$ is defined by $[a]_\rho * [b]_\rho = [ab]_p, \forall a, b \in S$.

Definition 1.6. [2, p.3] An alphabet is a finite set whose elements are letters. A word (over the alphabet A) is a finite sequence $u = (a_1, a_2, \dots, a_n)$ of letters of A. The integer n is the length of the word and is denoted |u|.

Definition1.7. [5, pg.43] Let A be alphabet and M^+ the set of all finite, nonempty words over A. Then M^+ is a free semigroup with respect to the operation defined as follows:
Given $(a_1 a_2 \dots a_n), (b_1 b_2 \dots b_n) \in M^+$ then $(a_1 a_2 \dots a_n), (b_1 b_2 \dots b_n) = (a_1 a_2 \dots a_m b_1 b_2 \dots b_n) \in M^+$.

Definition 1.8. [2, pg.29]. A semigroup presentation is a pair $P = (X, R)$ consisting of a set X and a set of pairs $R \subseteq X^+ x X^+$. A semigroup is defined by the presentation P if it is isomorphic to X^+/R, i.e. the quotient of the free semigroup X^+ by the least congruence containing all the pairs in R.

Definition 1.9. [2, pg.30] A semigroup presentation $\langle X \mid R \rangle$ is finite if X and R are finite. A semigroup is finitely presented if there exists some finite presentation that defines it, i.e. if it is isomorphic to $X^+|R$ for some finite presentation $\langle X \mid R \rangle$.

A **finitely presented semigroup** (resp. **finitely presented monoid**) is a quotient of a free semigroup (resp. free monoid) on a finite number of generators over a finitely generated congruence on the free semigroup (resp. free monoid).,

Proposition 1.10.[1, pg.3] Let $\langle A | R \rangle$ be a set and let S be any semigroup. Then any mapping $\phi : A \to S$ can be extended in a unique to a homomorphism $\bar{\phi} : A^+ \to S$ and A^+ is determined up to isomorphism by these properties.

We say that A^+ is the free semigroup on A.

Proposition 1.11. [1, pg.3] Every(finitely generated) semigroup is a homomorphism image of a (finitely generated) free semigroup.

Proposition 1.12. [1, pg.7] Let S be a semigroup and let A and B be two finite generating sets for S . If S can be defined by a finite presentation in terms of generators A, then S can be defined by a finite presentations in terms of generators B as well.

Proposition 1.13. [1, pg.3] Let S be a semigroup, let A be a generating set for S and let $R \subseteq A^+ x A^+$ then $\langle A | R \rangle$ is a presentation for S if and only if the following two conditions are satisfied:

a) S satisfies all the relations from R

b) if $u, v \in A^+$ are two words such that satisfies the relation $u = v$ then $u = v$ is a consequence of R.

2 . Semigroup Presentations

Lemma 2.1. Let S be a semigroup defined by the presentation
$P = \langle a, b : ab^2 = ba^2, a^{2n+1} = a, b^{2n+1} = b \rangle$. Then

a) $ab = ba^2 b^{2k}$ and $ba = ab^2 a^{2k}$;

b) $ab = (ba)^{k+1} a$ and $ba = (ab)^{k+1} b$;

c) $(ba)^{k+1} = aba^{2k}$ and $(ab)^{k+1} = bab^{2k}$;

d) $ab = aba^{2k+1}$ and $ba = bab^{2k+1}$;

Proof

a) $ba^2 b^{2k} = ab^2 b^{2k} = ab^{n+1} = ab$.

b) Prove that $ab = (ba)^r ab^{2k-2r+2}$ for $1 \le r \le k$.

$r = 1: ab = ba^2 b^{2k} = (ba) ab^{2k-2+2}$

$r \ge 1$: Suppose that $1 \le r \le k-1$ and the assertion holds for r . We have $2k - 2r > 0$.

$ab = (ba)^r ab^2 b^{2k-2r} = (ba)^r ba^2 b^{2k-2r} = (ba)^{r+1} ab^{2k-2(r+1)+2}$.

If we take $r = k$ we get $ab = (ba)^k ab^2 = (ba)^k ba^2 = (ba)^{k+1} a$.

c) Prove that $(ba)^{k+1} = (ba)^{k-r+1} ab^{2r} a^{2k}$ for $1 \le r \le k$.

$r = 1: (ba)^{k+1} = (ba)^k ba = (ba)^k ab^2 a^{2k}$ by (a).

$r \ge 1$: Suppose that $1 \le r \le k-1$ and the assertion holds for r.

$(ba)^{k+1} = (ba)^{k-r+1} ab^{2r} a^{2k} = (ba)^{k-r} ba^2 b^{2r} a^{2k} = (ba)^{k-r} ab^2 b^{2r} a^{2k} = (ba)^{k-(r+1)+1} ab^{2(r+1)} a^{2k}$.

If we take $k = r$ we obtain $(ba)^{k+1} = ba^2 b^{2k} a^{2k} = ab^2 b^{2k} a^{2k} = aba^{2k}$.

d) By b) and c) we get $ab = (ba)^{k+1} a = (aba^{2k}) a = aba^{2k+1}$

Theorem 2.2. Let S be a semigroup defined by the presentation $P = \langle a, b : ab^2 = ba^2, a^{2n+1} = a, b^{2n+1} = b \rangle$. Then

a) $|S| = \begin{cases} 10 & if \quad n = 1 \\ 68 & if \quad n = 2 \\ \infty & if \quad n \ge 3 \end{cases}$

b) $G = G_p$ has order 2 for n = 1 and $G = G_p$ has order 20 for n=2.

Proof

a)

```
gap> F := FreeSemigroup("a","b");
<free semigroup on the generators [ a, b ]>
gap> a := F.1;; b := F.2;;
gap> rels:=[[a^3,a],[b^3,b],[a*b^2,b*a^2]];
[ [ a^3, a ], [ b^3, b ], [ a*b^2, b*a^2 ] ]
gap> S := F/rels;
<fp semigroup on the generators [ a, b ]>
gap> Size(S);
10
gap> AsList(S);
[ a, b, a^2, a*b, b*a, b^2, a^2*b, a*b*a, a*b^2, a^2*b^2 ].

gap> F := FreeSemigroup("a","b");
<free semigroup on the generators [ a, b ]>
gap> a := F.1;; b := F.2;;
gap> rels:=[[a^5,a],[b^5,b],[a*b^2,b*a^2]];
[ [ a^5, a ], [ b^5, b ], [ a*b^2, b*a^2 ] ]
gap> S := F/rels;
<fp semigroup on the generators [ a, b ]>
gap> Size(S);
68
gap> AsList(S);
[ a, b, a^2, a*b, b*a, b^2, a^3, a^2*b, a*b*a, a*b^2, b*a*b, b^2*a, b^3, a^4, a^3*b,
a^2*b*a, a^2*b^2, (a*b)^2, a*b^2*a, a*b^3, (b*a)^2, b*a*b^2, b^2*a*b, b^3*a, b^4,
a^4*b, a^3*b*a, a^3*b^2, a*(a*b)^2, a^2*b^2*a, a^2*b^3, (a*b)^2*a, (a*b)^2*b,
a*b^2*a*b, a*b^3*a, a*b^4, b*a*b^3, b*(b*a)^2, b^2*a*b^2, a^4*b^2, a^2*(a*b)^2,
a^3*b^2*a, a^3*b^3, a*(a*b)^2*a, a*(a*b)^2*b, a^2*b^2*a*b, a^2*b^3*a, a^2*b^4,
(a*b)^2*b^2, a*b*(b*a)^2, (a*b^2)^2, b*a*b^4, a^4*b^3, a^2*(a*b)^2*a,
a^2*(a*b)^2*b, a^3*b^2*a*b, a^3*b^3*a, a^3*b^4, a*(a*b)^2*b^2, a^2*b*(b*a)^2,
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a*(a*b^2)^2, (a*b)^2*b^3, a^4*b^4, a^2*(a*b)^2*b^2, a^3*b*(b*a)^2, a^2*(a*b^2)^2,
a*(a*b)^2*b^3, a^2*(a*b)^2*b^3 ].
```

b)

```
gap>  F := FreeGroup("a","b");
<free group on the generators [ a, b ]>
gap>  a := F.1;; b := F.2;;
gap>  rels:=[[a^3,a],[b^3,b],[a*b^2,b*a^2]];
[ [ a^3, a ], [ b^3, b ], [ a*b^2, b*a^2 ] ]
gap> G := F/rels;
<fp group on the generators [ a, b ]>
gap>  Size(G);
2
gap> AsList(S);
gap> AsList(G);
[ <identity ...>, a ]
```

```
gap> F := FreeGroup("a","b");
<free group on the generators [ a, b ]>
gap> a := F.1;; b := F.2;;
gap> rels:=[[a^5,a],[b^5,b],[a*b^2,b*a^2]];
[ [ a^5, a ], [ b^5, b ], [ a*b^2, b*a^2 ] ]
gap>  G := F/rels;
<fp group on the generators [ a, b ]>
gap> Size(G);
20
gap> AsList(S);
gap> AsList(G);
[ <identity ...>, a, a^-1, b, b^-1, a^2, a*b, a*b^-1, a^-1*b,
a^-1*b^-1, b*a^-1, b^2, b^-1*a, a^2*b, a^2*b^-1, a*b*a^-1, a*b^2,
a*b^-1*a, a^-1*b*a^-1, a^2*b*a^-1 ]
```

Theorem 2.3. The semigroup $A = sgp(a,b,c; a = bab, b = cbc, c = aca)$ and it's corresponding group $A^* = gp(a,b,c; a = bab, b = cbc, c = aca)$ coincide.

Proof

$a = bab = bacbc = bacbaca$

=>

$bacbac = e$.

Then

$a = ea$

$c = aca = eaca = ec,$

$b = cbc = ecbc = eb.$

So e is a left neutral element. By symmetry, the element $cabcab$ is a right neutral element. But they coincide and are a unit element. Thus A has a unit element e.

Moreover, using the obvious symmetry of the presentation of A again, we see that

$e = (bac)^2 = (cab)^2 = (acb)^2 = (bca)^2 = (cba)^2 = (abc)^2$

$A_3^* = gp(a,b,c; a = bab, b = cbc, c = aca)$

Thus a has a left inverse, namely $cbacb$ and also a right inverse, $bcabc$. Correspondingly b and c have inverses and A is a group. It follows that semigroup A coincides with the group A^*.

```
gap> F := FreeGroup("a","b","c");
<free group on the generators [ a, b, c ]>
gap>  a := F.1;; b := F.2;; c:=F.2;;
gap> rels:=[[a,b*a*b],[b,c*b*c],[c,a*c*a]];
[ [ a, b*a*b ], [ b, b^3 ], [ b, a*b*a ] ]
gap> G := F/rels;
<fp group on the generators [ a, b, c ]>
gap> Size(G);
infinity
```

References

[1]. Campbell, C. M., Mitchell, J. D., & Ruškuc, N. (2002). "Comparing semigroup and monoid presentations for finite monoids". Monatshefte für Mathematik, 134(4), 287-293.

[2]. Torpey, M.(2019). "Semigroup congruences: computational techniques and theoretical Applications" (Doctoral dissertation, University of St Andrews).

[3]. Petrich, M. (1984). "Inverse semigroups" (No. 1). Wiley-Interscience.

[4]. Froidure, V., & Pin, J. E. (1997). "Algorithms for computing finite semigroups".

[5]. Nwawuru Francis & Udoaka, Otobong Gabriel.(2018). "Free semigroup presentations", Vol. 4 No. 3.