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SOME OSCILLATION CRITERIA FOR SECOND-ORDER NON-LINEAR DIFFERENTIAL EQUATIONS

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Abstract

We present new oscillation criteria for certain non-linear differential equations of second order with damping term

$$(r(t)x'(t))'+p(t)x'(t)+q(t)f(x(t))=0 \qquad t\geq t_0 \tag{1}$$

to used the oscillatory solutions of differential equations

$$(\alpha(t)x'(t))'+\beta(t)f(x(t))=0 \tag{2}$$

and

$$\left(a(t)x'(t)\right)'+\alpha(t)f\left(x\big(g(t)\big)\right)=0 \tag{3}$$

where that are different from most known ones (see [5], [8], [12], ect.). Our results extend and improve some previous oscillation criteria and cover the cases which are not covered by known results. In this paper, by using the generalized Riccati technique we get a new oscillation and non-oscillation criteria for (1). The theorems prove to be efficient in many cases and give the results in the literature.

Keywords: differential, equations, interval, criteria, damping, second order

1. Introduction

In this paper we are being consider the oscillation solutions in the second order nonlinear functional differential equation:

$$(r(t)x'(t))'+q(t)x'(t)+p(t)f(x(t))=0\,, \qquad t\geq t_0 \tag{1}$$

where we very often use the following assumptions :

A1) $p(t)$ is a real valued and locally integrable function over $I=[\alpha,\infty)$ and not identically zero in any neighborhood of ∞.

A2) $q(t)$ is a real valued and locally integrable over I .

A3) For all $t\in I, r(t)>0$, for $t\in I=[\alpha,\infty)$, and $\int_\alpha^\infty \frac{1}{r(t)}dt=\infty$.

A4) $f\in C(R,R)$, $xf(x)>0$, and $f'(x)\geq k>0$.

By a solution of equation (1) or (2) we consider a function $x(t), t\in[t_x,\infty)\subset[t_0,\infty\}$ which is twice continuously differentiable and satisfies equation (1) or (2) on the given interval. The number depends on that particular solution x(t) under consideration.

We consider only non-trivial solutions. A solution x(t) of (1) or (2) is said to be oscillatory if there exists a sequence $\{\lambda_n\}_{n=1}^\infty$ of points in the interval $[t_0,\infty\}$, so that $\lim_{n\to\infty}\lambda_n=\infty$ and $x(\lambda_n)=0, n\in N$, otherwise it is said to be non-oscillatory. An equation is said to be oscillatory if all its solutions are oscillatory, otherwise it is considered that is non-oscillatory solution .

An important tool in the study of oscillatory behavior of solutions of these equations is the averaging technique . We can se in this paper that the equation

$$(a(t)x'(t))'+p(t)f(x(t))=0 \tag{2}$$

may be transform in the form (1) and the solution of (2) are also solution of (1).
Leighton's result (see [9]) where every solution of (2) when f(x)=x is oscillatory need to satisfy the conditions

$$\int_0^\infty \frac{1}{a(t)}dt=\infty$$

and

$$\int_0^\infty p(t)=\infty .$$

For the same case Willet in [10] obtain the following criteria : If

$$\int_0^\infty \frac{1}{a(t)}dt<\infty$$

and

$$\int_{t_0}^\infty a(t)\left(\int_t^\infty \frac{ds}{a(s)}\right)^2 dt=\infty$$

then every solution of (2) oscillates.
N. Yamaoka in [11] presented the result : If A2 , A3 hold and a(t), b(t) satisfy

$$a(t)b(t)\left(\int_t^\infty \frac{1}{a(l)}dl\right)^2 \geq 1$$

for t sufficiently small, and that there exists a $\lambda > \frac{1}{16}$, such that

$$\frac{f(x)}{x} \geq \frac{1}{4}+\frac{\lambda}{(\log|x|)^2}$$

for $|x|$ sufficiently small. Then all solutions of (2) are oscillatory.
Moreover equation include the differential equations where f(x)=x which recently had been discussed by R. Kim on [3].
On the continuation we present the well – known Gronwall's inequality.

Lemma 1. Let $I=[t_0,T)$, be an interval of real numbers , and suppose that

$$u(t) \leq c+\int_{t_0}^t q(s)u(s)ds, \text{ for } t\in I , \tag{4}$$

where c is a nonnegative constant , and $u,q\in C(I,\Re^+)$. Then ,

$$u(t) \leq c \, exp \int_{t_0}^t q(s)ds, \text{ for } t\in I .$$

The Conditions for oscillatory solutions of the second order differential equations (1) are studied by many authors (see [2], [4], [6],[7] etc.). Here , we give some conditions for coefficients where the equation (1) have oscillatory solutions and we also take in to account the result we have obtained in the previous researches , here we present more generalized criteria that define oscillation solution of the equation (1) to used oscillatory solutions of (2).
In this paper are presented theorems, that use generalized Riccati – type transformations, and averaging

technique , which explain results for oscillatory nature of differential equations also, our results extend and improve a number of existing results (see [3], [8], etc.).

2. Main result

What follows is, $E(l)$, $\beta(l)$ denote

$$E(l)=e^{\int_{\alpha}^{t}\frac{q(l)}{r(l)}dl}$$

and

$$B(\alpha)=\int_{\alpha}^{\infty}(p(l)-\frac{q^{2}(l)}{4kr(l)})dl$$

that we will use in the following theorem.

Theorem 1 : The equation (1) is oscillatory if for $p(t)\geq 0$, $t\geq\alpha$, and

$$\int_{\alpha}^{\infty}\frac{1}{E(l)r(l)}dl=\infty \tag{2}$$

$$B(\alpha)=\infty \tag{3}$$

Proof: For

$$E(l)=e^{\int_{\alpha}^{t}\frac{q(l)}{r(l)}dl},$$

we have

$$E'(t)=\frac{q(t)}{r(t)}E(t),$$

where $W(t)>0$, and the equation (1) is reduced in to

$$(E(t)r(t)x'(t))'+E(t)p(t)f(x(t))=0 \tag{4}$$

We se that (1) is oscillatory if and only if the equation (4) is oscillatory.
Assume that (1) is non-oscillatory. Then there exists a non-oscillatory solution *x(t)* of (1) . So we may assume that $x(t)>0$ on $[t_1,\infty)$, for some $t_1>\alpha$. We show that $x'(t)>0$, for $t\geq t_1$.
From (4) we obtain that

$$(E(t)r(t)x'(t))'=-E(t)p(t)x(t)\leq 0$$

from what $E(t)r(t)x'(t)$ is not increasing for $t\geq t_1$. Assume that $E(t_2)r(t_2)x'(t_2)<0$ for some $t_2>t_1$.
Put $E(t_2)r(t_2)x'(t_2)=L$, then for $t\geq t_2$, we have

$$E(t)r(t)x'(t)\leq L.$$

Dividing both sides by $E(t)r(t)$ and integrating from t_2 to t $(>t_2)$, we obtain

$$x(t)-x(t_2)\leq L\int_{t_2}^{t}\frac{1}{E(l)r(l)}dl$$

because $L\int_{t_2}^{t}\frac{1}{E(l)r(l)}dl$ is tending to $-\infty$, where $t\to\infty$, we conclude that $x(t)<0$, for sufficiently large t, which is a contradiction . Therefore $x'(t)>0$, for $t\geq t_1$.
In case that $x(t)<0$, put $y(t)=-x(t)$. So we have $x'(t)>0$.

Considering the function

$$W(t) = \frac{r(t)x(t)}{f(x(t))}$$

we have

$$W'(t) = -\frac{q(t)}{r(t)}W(t) - p(t) - \frac{W^2 f'(x(t))}{r(t)}$$

and from $f'(x) \ge k > 0$, we obtain

$$W'(t) \le -\frac{q(t)}{r(t)}W(t) - p(t) - \frac{W^2(t)k}{r(t)},$$

$$W'(t) \le -\frac{1}{r(t)}(W(t)\sqrt{k} + \frac{q(t)}{2\sqrt{k}})^2 + \frac{q^2(t)}{4kr(t)} - p(t) \qquad (5)$$

Integrating (5) from t_2 to t $(> t_2)$, we get

$$W(t) - W(t_1) + \int_{t_1}^{t} (p(l) - \frac{q^2(l)}{4kr(l)})dl \le -\int_{t_1}^{t} \frac{1}{r(l)}(W(l)\sqrt{k} - \frac{q(l)}{2\sqrt{k}})^2 dl \,.$$

By means of (4) there exists a $t_3 \ge t_1$, such that for $t \ge t_3$, we gain

$$W(t) \le -\int_{t_1}^{t} \frac{1}{r(l)}(W(l)\sqrt{k} - \frac{q(l)}{2\sqrt{k}})^2 dl$$

which is impossible because $W(t) > 0$, for $t \ge t_1$.

Lemma 1: Assume that for $t \ge \alpha$, $p(t) \ge 0$, $q(t) \ge 0$ and (3) are valid. If the differential equations (1) has a positive solution , we have

$$\lim_{t\to\infty} \frac{r(t)x'(t)}{f(x(t))} = 0.$$

Proof : Let $x(t) > 0$, be a solution of (1) . From Theorem 1. it follows that from $p(t) \ge 0$, for $t \ge \alpha$, and (3) that there exists a $t_1 \ge \alpha$ such that $x'(t) > 0$, for $t \ge t_1$.
Put

$$W(t) = \frac{r(t)x'(t)}{f(x(t))} > 0$$

for $t \ge t_1$, and consider Ricati inequation

$$W'(t) \le -\frac{q(t)}{r(t)}W(t) - p(t) - \frac{W^2(t)k}{r(t)}$$

it is obvious that

$$-\frac{W'(t)}{W(t)} \ge \frac{k}{r(t)}.$$

Integrating the above inequation over $[t_1, \infty)$ and considering the condition $r(t) > 0$, and

$$\int_{\alpha}^{\infty} \frac{1}{r(t)} dt = \infty,$$

we have

$$-\int_{\alpha}^{x}\frac{W'(t)}{W(t)}dt=\int_{\alpha}^{x}\frac{1}{r(t)}dt$$

$$\frac{1}{W(x)}=\frac{1}{W(\alpha)}+\int_{\alpha}^{x}\frac{1}{r(t)}dt$$

from where for $x\to\infty$, we obtain

$$\lim_{x\to\infty}W(x)=0.$$

To used the result presented in [8] , here in follow we give the bounded solution of (2) with :

Theorem 2. Suppose $\alpha(t),\beta(t)\in C^1([t_0,\infty))$ and $(\alpha(t)\beta(t))'\geq 0$, for $t\geq t_0$ and

$$\int^{\pm\infty} f(l)dl=\infty, \qquad (6)$$

then the solution $x(t)$ of the equation (2) such that $x(t_1)=0,\ t_1\geq t_0$ for some $t_1\in[t_0,\infty)$ is bounded.

Proof : Let $x(t)$ be an arbitrary solution of equation (1) such that $x(t_1)=0,\ t_1\geq t_0$.

Put

$$F(s)=\int_{x(t_1)}^{s} f(s)dl. \qquad (7)$$

Multiplying the equation (2) by $\alpha(t)x'(t)$, we gain

$$\frac{1}{2}((\alpha(t)x'(t))^2)'+\alpha(t)x'(t)\beta(t)f(x(t))=0$$

integrating from t_1 to t, we obtain

$$\frac{1}{2}(\alpha(t)x'(t))^2-\frac{1}{2}(\alpha(t_1)x'(t_1))^2+$$

$$+\alpha(t)\beta(t)F(x(t))-\int_{t_1}^{t}(\alpha(s)\beta(s))'F(x(s))ds=0. \qquad (8)$$

Denote

$$c=\frac{1}{2}(\alpha(t_1)x'(t_1))^2$$

now by (8) it follows that :

$$\alpha(t)\beta(t)F(x(t)\leq c+\int_{t_1}^{t}(\alpha(s)\beta(s))'F(x(s))ds=c+\int_{t_1}^{t}\frac{(\alpha(s)\beta(s))'}{\alpha(s)\beta(s)}\alpha(s)\beta(s)F(x(s))ds. \quad (9)$$

Hence , by Granwalls inequality , we get

$$\alpha(t)\beta(t)F(x(t)\leq c\cdot e^{\int_{t_1}^{t}\frac{(\alpha(s)\beta(s))'}{\alpha(s)\beta(s)}ds} \qquad (10)$$

from this

$$F(x(t))\leq\frac{c}{\alpha(t_1)\beta(t_1)},\ \ t_1>t_0>0 \qquad (11)$$

so, $F(x(t))$ is bounded and from (6) the solution $x(t)$ is bounded .

Example: Consider differential equation

$$(\frac{1}{e^t}x'(t))' + \frac{1}{t^2 \ln t}x'(t) + \frac{1}{1+e^t}(x(t)+x^3(t)) = 0,\ \ t>0 \qquad (12)$$

$$\text{for}\ \ r(t) = \frac{1}{e^t},\ q(t) = \frac{2}{1+e^t}\ \text{ and }\ f(x) = x + x^3,$$

from where $f'(x) = 1 + 3x^2 \geq 1 = k > 0$.

Also , for

$$E(l) = e^{\int_{\alpha}^{l} \frac{e^t}{1+e^t}dt} = e^t - e^{\alpha},$$

we have

$$\int_{\alpha}^{\infty} \frac{1}{E(l)r(l)} dl = \infty$$

and

$$B(\alpha) = \int_{\alpha}^{\infty} (p(l) - \frac{q^2(l)}{4kr(l)})dl = \int_{\alpha}^{\infty} (\frac{1}{t\ln t} - \frac{e^t}{4k(1+e^t)^2})dt = \infty$$

that from the theorem 1. the equation (12) is oscillatory.
In the same way may try to end [3]() (see [13])

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References

[1]. H. El – Metwally, M. A. El –Moneam, The oscillatory behavior of second order nonlinear functional differential equations, the Arabian Journal for science and engineering, volume 31. Number 1A,23 – 30, January 2006.

[2]. Samir H. Saker , Oscillation theorems for second order nonlinear functional differential equations with damping, dynamic sys. appl. 12 , 2003, 307 - 322

[3]. Rak Joong Kim, Oscillation and non-oscillation criteria for differential equations of second order, Korean J. Math. 20011, N0.4, pp. 391 – 402

[4]. Xh. Beqiri, E. Koci; Interval oscillation criteria for second order nonlinear differential equations with damping term, Inter. Jurn. of science, innovat. and new technol., Vol.1. N.2, 85 – 92 ,2011,

[5]. Aydin Tiryaki; Oscillation criteria for a certain second order nonlinear differential equations with deviating arguments; ; electronic journal of qualitative theory of differential equations, 2009, N0 . 61,1-11

[6]. Yuri V. Rogovchenko; Interval oscillation of a second order nonlinear differential equation with a damping term, Discrete and continuous dynamical systems supplement , pp. 883 – 891, 2007

[7]. M.M.A. El Sheikh, R.A.Sallam, D.I. Elimy; oscillation criteria for nonlinear second order damped differential equations, International journal of nonlinear science , vol.10, nr. 3, 297 – 307, 2010.

[8]. J. Ohriska, A. Zulova, Oscillation criteria for second order non – linear differential equations, , Institute of mathematics , P. J. Safarik University, Jesenna,

[9]. p. 1 – 102004.

[10]. W. Leighton, The detection of the oscillation of solution a second order differential equation, Duke Math. J. 17 (1950), 57 – 62.

[11]. D. W. Willett, On the oscillatory behavior of the solutions of second order linear

[12]. differential equations, Ann. Polon. Math. 21 (1969), 175—191.

[13]. N. Yamaoka, Oscillation and non-oscillation theorems for second order nonlinear differential equations with p – Laplacian, Math. science, 38 (2005) , pp. 17 – 30.
[14]. Xh. Beqiri, E. Koci, new oscillation and non oscillation criteria for second order nonlinear differential equations. International journal of pure and applied mathematics, (2015) pp, 158 – 163.
[15]. Xh. Beqiri, E. Koci, interval oscillation criteria for second order nonlinear differential equations, International konference , advanced studies in mathematic p. 38, 2011, Tirana.