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A PREDICTOR-CORRECTOR METHOD FOR SOLVING DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER

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Abstract

The story of the fractional calculus began with that letter in 1695, answered by Leibniz. Classical analysis is not always sufficient, for real problems, constructed by using mathematical expressions, for solving their applications in engineering, science, and many other fields. The aim of this paper is to show the effectiveness of the numerical predictor-corrector method known as Fractional Adams-Bashforth-Moulton method (FABM) by its application on solving different types of nonlinear differential equations of fractional order $0 < \alpha < 1$. It contains a short survey of basic numerical method (FABM) using the fractional derivative defined by Caputo. The equivalence between an ordinary differential equation of fractional order and a suitable Volterra integral equation is key to the approaches. The numerical results for the constructed method are compared with the exact solution for each equation by using absolute error (absolute difference between the exact and approximate solution at each integration point).The method is very simple and very much effective for solving differential equations of fractional order, it may be used. The behavior of the approximate time-series solutions are tabulated and plotted at different values of the fractional orders. During the work, it became necessary to use such symbolic software packages as Mathematica 12.1 in completing the required steps of the above procedures.

Keywords: Fractional initial-value problem, Caputo fractional derivative, Volterra equation, Fractional Adams-Bashforth-Moulton method, Exact solution.

1. Introduction

Fractional calculus tools have been known and used in different fields for a long time, the theory of fractional differential equations has recently begun to be studied. Differential equations of fractional order, have gained interest in many different scientific areas, especially in engineering real problems. As most of fractional differential equations do not have analytic solutions, we have to use methods to convert them to more accurate equations, like Volterra integral equation, for which we can then use various approximation and numerical techniques [1,2,3]. Several real-world phenomena in physics, engineering and science fields can be demonstrated successfully by developing a model using the theory of fractional calculus. Numerous problems in Physics, Chemistry, Engineering and Biological Sciences are better described in terms of differential equations of fractional order. The exact or semi-analytical solution to many physical problems can be understood by studying a physical phenomenon's future, current and historical states.

We use the Adams-Bashforth-Moulton method [4,5,7] to find approximated solutions to different equations, with or without exact solution. This is a well-known numerical method developed initially for solving ordinary differential equations of first order. After the modifications, it is a good scheme for solving differential equations of fractional order. It proceeds in two steps. Firstly, the prediction step calculates a rough approximation of the desired solution [6]. Secondly, the corrector step refines the initial approximation using another means.

Definition 1.1.[1,2,9] A real function $f(t),\ t>0$, is said to be in space $C_\mu\,,\mu\in\mathbb{R}$, if there exist a real number $p(>\mu)$, such that $f(t)=t^p f_1(t)$, where $f_1(t)\in C[0,\infty)$, and it is said to be in the space C_μ^m if $f^{(m)}\in C_\mu\,,\mu\in\mathbb{N}$.

Definition 1.2.[1,2,9] The Riemann-Liouville fractional integral operator of order $\alpha\geq 0$ of a function $f(t)\in C_\mu\,,\ \mu\geq -1$ is defined as:

$$J^\alpha f(t)=\frac{1}{\Gamma\alpha}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau \tag{1.1}$$

Definition 1.3. [9] The Caputo fractional derivative of a function $f(t)$ is defined by:

$$D_t^\alpha f(t)=J^{n-\alpha}D_t^n f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)d\tau \tag{1.2}$$

Two basic properties of Caputo fractional derivative that immediately follow from Definition 1.3 are:

i. $J^\alpha D_t^\alpha f(t)=f(t)-\sum_{k=0}^{n-1} f^k(0^+)\frac{x^k}{k!},t>0,$

ii. $D_t^\alpha J^\alpha f(t)=f(t).$

Remark 1. In the definition for Caputo fractional derivative, we first differentiate $f(t), n-$times, then integrate it $(n-\alpha)$ times. If $f(t)$is $n-$times differentiable, then the $\alpha-th$ order derivative will exist, where $n-\alpha<\alpha\leq n$, otherwise this definition is not applicable [2,9]. Two main advantages of this definition are:

i. Fractional derivative of a constant is zero,
ii. Fractional differential equation of Caputo type has initial conditions of classical non-integer derivative type, in contrast to fractional differential equation of Riemann-Liouville type, where initial conditions are of fractional type.

2. Numerical method

Consider the initial value problem

$$D_t^\alpha y(t)=f\big(t,y(t)\big),\quad y^{(k)}(0)=y_0^{(k)} \tag{2.1}$$

$(k=0,1,2,\dots,\lceil\alpha\rceil-1)$, we will construct the numerical method of Adams-Bashforth-Moulton (FABM), assuming that a solution of (2.1) is sought on some time interval $[0,T]$, arbitrary $0<\alpha<n$ and $f:[0,T]\times D\to\mathbb{R}, D\subseteq\mathbb{R}$. The interval $[0,T]$ is divided into l subintervals [1,2,3]. Consider an equi-spaced grid with step length h, $t_j=jh,\ \ j=0,1,\dots$ .Let y_j denote the approximated solution at t_j and $y(t_j)$ denote the exact solution of the initial value problem (2.1).

2.1. Fractional Adams-Bashforth-Moulton Method

In order to assure the existence and uniqueness of the solution to (2.1), it is assumed that $f\big(t,y(t)\big)$ is continuous and fulfills the Lipschitz condition with respect to the second variable. On $[0,T]$, for a uniform grid $t_j=hj$ $(j=0,1,\dots,N)$ and a constant time step denoted by $h=\frac{T}{N}$, the goal is to approximate solution values $y_j\approx y\big(t_j\big)$ at the grid points [3,4,7].

According to the theorem of existence and uniqueness of the solution, initial value problem (2.1) can be reformulated in terms of the weakly-singular Volterra integral equation at the point t_n:

$$y(t_n)=\sum_{k=0}^{\lceil\alpha\rceil-1}\frac{y_0^{(k)}}{k!}t^k+\frac{1}{\Gamma(\alpha)}\int_0^{t_n}(t_n-\tau)^{\alpha-1}f(\tau,y(\tau))d\tau \tag{2.2}$$

The method immediately suggests a numerical approach in solving (2.2). On each subinterval $[t_k,t_{k+1}]$, $k=0,\dots,n-1$, the function $f(t,y(t))$is approximated by constant value $f(t_k,y(t_{k+1}))$, using the fractional rectangular formula to obtain the main part of the algorithm, known as Adams-Bashforth part (FAB)[4,7]:

$$y[j]=\sum_{k=0}^{\lceil\alpha\rceil-1}\frac{(jh)^k}{k!}y_0^{(k)}+\frac{h^\alpha}{\Gamma(\alpha+1)}\sum_{k=0}^{j-1}b[j-k]f(kh,y[k]) \tag{2.3}$$

where $b[j-k]$ are the weights which depend only on the difference $(j-k)$ because of the convolution structure of $b_{k,j}$:

$$b_{k,j}=\frac{(j-k)^{\alpha}-(j-k-1)^{\alpha}}{\Gamma(\alpha+1)} \tag{2.4}$$

(FAB) part is a natural candidate for a predictor in the process of constructing the predictor- corrector method FABM (the Adams-Moulton method can be constructed in similar way like FAB).
The FABM method is said to be Predict-Evaluate-Correct-Evaluate type because an initial approximation p, the so-called predictor, is evaluated first:

$$p=\sum_{k=0}^{\lceil\alpha\rceil-1}\frac{(jh)^{k}}{k!}y_0^{(k)}+\frac{h^{\alpha}}{\Gamma(\alpha+1)}\sum_{k=0}^{j-1}b[j-k]f(kh,y[k]) \tag{2.5}$$

Then the method gives the corrector formula:

$$y[j]=\sum_{k=0}^{\lceil\alpha\rceil-1}\frac{(jh)^{k}}{k!}y_0^{(k)}+\frac{h^{\alpha}}{\Gamma(\alpha+2)}\sum_{k=0}^{j-1}\left(f(jh,p)+(j-1)^{\alpha+1}-(j-\alpha-1)j^{\alpha}\cdot f(0,y[0])+\sum_{k=0}^{j-1}a\,[j-k]f(kh,y[k])\right) \tag{2.6}$$

where p represents FAB, which in this case acts like a predictor. The weight $a[j-k]$ in the corrector $y[j]$is given by

$$a[j-k]=\begin{cases}\frac{(j-1)^{\alpha+1}-(j-\alpha-1)j^{\alpha}}{\Gamma(\alpha+2)},\ if\ j=0\\ \frac{(j-k+1)^{\alpha+1}+(j-k-1)^{\alpha+1}-2(j-k)^{\alpha+1}}{\Gamma(\alpha+2)},if\ j\in[1,k-1]\\ 1,\ if\ j=k\end{cases} \tag{2.7}$$

For $0<\alpha<1$, denoting (2.5) with $p\equiv y^{p}[j]$ and (2.6) with $y[j]\equiv y^{c}[j]$, the fractional Adams–Bashforth–Moulton (FABM) formula is:

$$y^{p}[j]=y_0+\frac{h^{\alpha}}{\Gamma(\alpha+1)}\sum_{k=0}^{j-1}b[j-k]f(kh,y[k])$$

$$y^{c}[j]=y_0+\frac{h^{\alpha}}{\Gamma(\alpha+2)}\sum_{k=0}^{j-1}\left(f(jh,y^{p}[j])+(j-1)^{\alpha+1}-(j-\alpha-1)j^{\alpha}\cdot f(0,y[0])\right.$$
$$\left.+\sum_{k=0}^{j-1}a\,[j-k]f(kh,y[k])\right) \tag{2.8}$$

3. Applications

Example 3.1. Let consider the differential equation of fractional order $0<\alpha<1$ [1]:

$$D^{\alpha}y(t)=t+y^{2},\ \ y(0)=0 \tag{3.1}$$

The equation (3.1) do not have the exact analytical solution. We will find the approximated ones using (FABM) for different values of $0<\alpha<1$, $\alpha=0.5$, $\alpha=0.7$ and $\alpha=0.9$. The exact solution of (3.1) exist for $\alpha=1$, the comparison of the method will be done at this case.Using (2.8) there are found the following approximated time series.

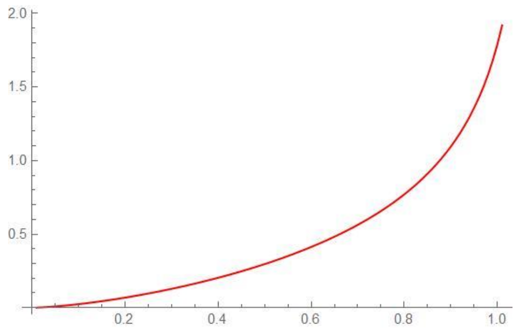

Fig 1. Time-series solution$y(t)$ versus $t$of (3.1) for $\alpha = 0.5$, $t \in [0,1]$, $h = 0.01$ using (FABM) method (red\solid line)

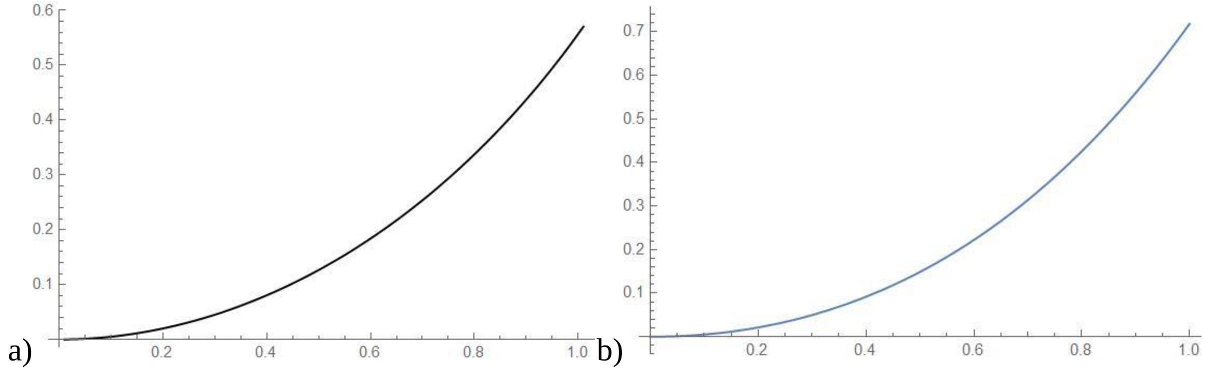

Fig 2. Time-series solution$y(t)$ versus t of (3.1) for $\alpha = 1$, $t \in [0,1]$,$h = 0.01$ a) using FABM (black/solid line), b) plotting the exact integration curve$y = -1 + e^t - t$ of (3.1) (blue/solid line)

The results obtained by (FABM) are shown for different values of $\alpha = 0.5$, $\alpha = 0.7$ and $\alpha = 0.9$ and compared for $\alpha = 1$with the exact solution using the absolute error.

Table 1. Partial data values of the time-series $y(t)$ for increasing t of the numerical approximated solutions of (3.1) using (FABM) and exact solution of (3.1) for standard form $\alpha = 1$, as well as the corresponding values of absolute error between the FABM and exact solution for $\alpha = 1$.

Time $t \in [0,1]$	**(FABM)**				**Exact solution for *v*=1**	**Absolute Error for *v*=1**
	v=0.5	*v=0.7*	*v=0.9*	*v=1*		
0.01	0.000752	0.000258	0.0000867	0.00005	0.0000502	0.0000002
0.03	0.00391	0.001668	0.0006994	0.00045	0.00045453	0.0000045
0.05	0.008417	0.003976	0.001846	0.00125001	0.0012711	0.0000211
0.07	0.013955	0.007047	0.0034986	0.00245008	0.00250818	0.0000581
0.09	0.02037	0.010806	0.0056403	0.00405028	0.00417428	0.0001240
0.11	0.027566	0.015206	0.0082593	0.00605078	0.00627807	0.0002273
0.13	0.035482	0.020212	0.0113464	0.00845182	0.00882838	0.0003766
0.15	0.044074	0.025797	0.0148948	0.0112537	0.0118342	0.0005805
0.17	0.05331	0.031941	0.0188988	0.014457	0.0153049	0.0008479
0.19	0.063169	0.038629	0.023354	0.0180623	0.0192496	0.0011873
0.21	0.073636	0.045848	0.0282571	0.0220703	0.0236781	0.0016078
0.23	0.084701	0.053589	0.0336053	0.026482	0.0286	0.0021180

0.25	0.096361	0.061846	0.0393969	0.0312987	0.0340254	0.0027267
0.27	0.108614	0.070613	0.0456308	0.0365216	0.0399645	0.0034429
0.29	0.121465	0.079888	0.0523065	0.0421525	0.0464275	0.0042750
0.3	0.128117	0.084715	0.0558101	0.0451215	0.0498588	0.0047373
…	…	…	…	…	…	…
0.9	1.09589	0.676774	0.499672	0.437472	0.559603	0.12213
0.91	1.14049	0.694813	0.512198	0.448484	0.574323	0.12584
0.93	1.24039	0.732398	0.537993	0.47111	0.604509	0.13340
0.95	1.35848	0.772159	0.564824	0.494571	0.63571	0.14114
0.97	1.50177	0.814321	0.59275	0.518905	0.667944	0.14904
0.99	1.68174	0.859146	0.621838	0.544153	0.701234	0.15708
1	1.79129	0.88265	0.636839	0.557134	0.718282	0.16115

In all numerical simulations, we take the integration step-size $h = 0.01$, and the integration time-span is $t \in [0, 1]$. It is seen that all the curves are characterized with a typical exponential-like increase as $t \to \infty$. The numerical methods gradually diverge from the exact solution as the time rises, and in this case (FABM) is slightly a good approximation at large time values.

Example 3.2. Let consider the inhomogeneous linear equation [8] for $0 < \alpha < 1$,

$$D^{\alpha} y(t) + y(t) = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} + t^2 - t\,,\ \ y(0) = 0 \qquad (3.2)$$

The exact solution of (3.2) using the analytical method of Laplace Transform is $y(t) = t^2 - t$.
The results are shown in Fig ure 3, for $\alpha = 0.95$, step size $h = 0.01$ and time $t \in [0,10]$.

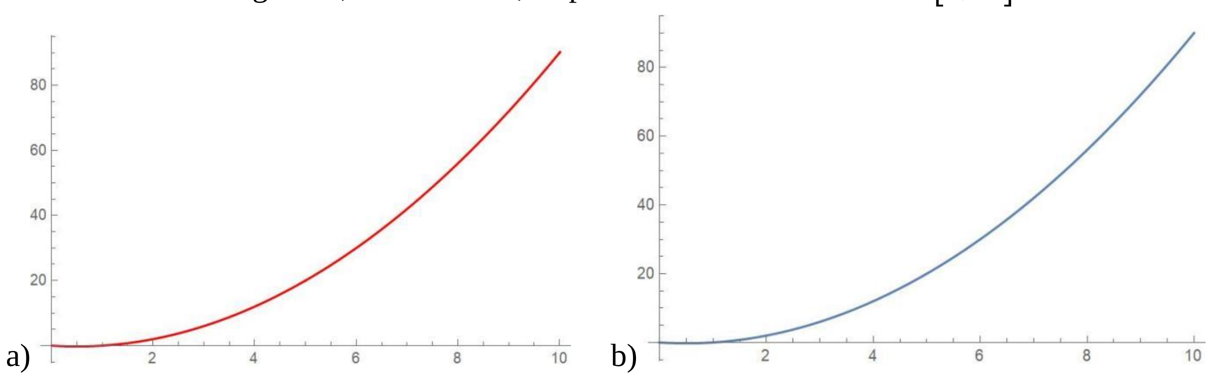

Fig 3. Time-series solution $y(t)$ versus t of (3.2) for $\alpha = 0.95$, $t \in [0,1]$, $h = 0.01$ a) using FABM (red/solid line), b) plotting the exact integration curve $y = t^2 - t$ of (3.2) (blue/solid line)

Table 2. Partial data values of the time-series $y(t)$ for increasing t of the numerical approximated solutions of (3.2) using (FABM) and exact solution of (3.2) for standard form $\alpha = 0.95$, as well as the corresponding values of absolute error between the FABM and exact solution for $\alpha = 0.95$.

Time $t \in [0, 10]$	FABM for $\alpha = 0.95$	Exact solution for $\alpha = 0.95$	Absolute error
0.01	-0.00533877	-0.0099	0.0045612
0.03	-0.0249456	-0.0291	0.0041544
0.05	-0.0435512	-0.0475	0.0039488
0.07	-0.0613081	-0.0651	0.0037919
0.09	-0.0782408	-0.0819	0.0036592
0.11	-0.0943585	-0.0979	0.0035415
0.13	-0.109666	-0.1131	0.0034340
0.15	-0.124165	-0.1275	0.0033350
0.17	-0.137857	-0.1411	0.0032430
0.19	-0.150745	-0.1539	0.0031550
0.21	-0.162827	-0.1659	0.0030730
0.23	-0.174106	-0.1771	0.0029940
0.25	-0.184582	-0.1875	0.0029180
0.27	-0.194254	-0.1971	0.0028460
0.29	-0.203123	-0.2059	0.0027770
0.31	-0.21119	-0.2139	0.0027100
0.33	-0.218454	-0.2211	0.0026460
0.35	-0.224916	-0.2275	0.0025840
0.37	-0.230576	-0.2331	0.0025240
0.39	-0.235434	-0.2379	0.0024660
0.41	-0.23949	-0.2419	0.0024100
0.43	-0.242745	-0.2451	0.0023550
0.44	-0.244071	-0.2464	0.0023290
0.45	-0.245198	-0.2475	0.0023020
0.47	-0.246849	-0.2491	0.0022510
0.49	-0.247698	-0.2499	0.0022020
…	…	…	…
9.51	80.9302	80.9301	0.0001000
9.53	81.291	81.2909	0.0001000
9.55	81.6526	81.6525	0.0001000
9.57	82.015	82.0149	0.0001000
9.59	82.3782	82.3781	0.0001000
9.61	82.7422	82.7421	0.0001000
9.63	83.107	83.1069	0.0001000
9.65	83.4726	83.4725	0.0001000
9.67	83.839	83.8389	0.0001000
9.69	84.2062	84.2061	0.0001000
9.71	84.5742	84.5741	0.0001000
9.73	84.943	84.9429	0.0001000

9.75	85.3126	85.3125	0.0001000
9.77	85.683	85.6829	0.0001000
9.79	86.0542	86.0541	0.0001000
9.81	86.4262	86.4261	0.0001000
9.83	86.799	86.7989	0.0001000
9.85	87.1726	87.1725	0.0001000
9.87	87.547	87.5469	0.0001000
9.89	87.9222	87.9221	0.0001000
9.91	88.2982	88.2981	0.0001000
9.93	88.675	88.6749	0.0001000
9.95	89.0526	89.0525	0.0001000
9.97	89.431	89.4309	0.0001000
9.99	89.8102	89.8101	0.0001000
10	90.0001	90	0.0001000

In this numerical simulation, we take the integration step-size $h = 0.01$, and the integration time-span is $t \in [0, 10]$. It is seen that all the curves are characterized with a typical exponential-like increase as $t \to \infty$. The numerical methods do not gradually diverge from the exact solution as the time rises, and in this case the absolute error become constant during the time.

4. Conclusions

Considerable attention is paid to differential equations of fractional order because they appear to be more effective for modeling and analyzing dynamical processes in basic and engineering and sciences. Numerical solutions of the equations are often the only approach to study the dynamical behavior of these at particular parameter values, since in general the exact solutions of differential equations of fractional-order cannot be sought in practice. In our current work, we aimed to present the general form of (FABM) and its application to numerically approximate two different linear differential equations of fractional order in which fractional derivatives are taken in the sense of Caputo. To characterize the successfulness of the applied numerical method, we use the absolute difference parameter between the exact solution of each equation at a given time point and the approximate solution at the same instance. From the resulting diagrams and tabular values, we conclude that, in the Example 3.1, the method gradually diverge from the exact solution as the time rises, but in the Example 3.2. it approaches the correct solution, until the error becomes constant. During the work, it became necessary to use such symbolic software packages as Mathematica 12.1 in completing the required steps of the above procedures.

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