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# THE "NE" INTEGRAL TRANSFORM AND FRACTIONAL DIFFERENTIAL EQUATIONS

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**Abstract**

This article demonstrates how the new Double "NE" integral transform is successfully implemented to obtain the exact solutions of fractional nonlinear partial differential equations by considering specified conditions. Several properties and theorems related to existing conditions, partial derivatives, the double convolution theorem, and others are presented. The new technique with double "NE" integral transform is efficient and accurate in examining exact solutions of FPDEs. To show the applicability of the presented method, some examples are illustrated.

*Keywords:* Double "NE" integral transform, fractional partial differential equations, "NE" integral transform, Caputo fractional derivatives

## 1. Introduction

Fractional partial differential equations appear in various applications of science, such as chemistry, physics, engineering, and mathematics, which is why researchers have established many techniques for solving such equations such as the homotopy perturbation method, variation iteration method, Adomian decomposition method, finite difference method, and others. The fractional calculus generalizes the operations of differentiation and integration to noninteger orders. Fractional calculus has become an important tool for the study of some physical phenomena, engineering, and science, such as electromagnetics, viscoelasticity, fluid mechanics, electrochemistry, biological population models, optics, and signal processing. Furthermore, fractional calculus processes have become one of the most useful approaches in a variety of applied sciences to deal with certain properties of (long) memory effects. There are many definitions of fractional derivatives, such as Riemann–Liouville, Caputo, Caputo–Fabrizio, Atangana–Baleanu, conformable and the generalized fractional derivative

The method of double integral transforms is a hot topic in recent research, and it depends on applying a single transformation twice on functions of two variables or applying two different transformations on the same function. This new approach is a powerful tool for solving PDEs. Although double integral transformations, their properties, and theorems are recent studies, they have attracted the interest of many mathematicians. Therefore, many researchers have studied new combinations, such as double Laplace transform, double Sumudu transform, double Elzaki transform, double Laplace-Sumudu transform and others.

Recently, a new integral transform, called the "NE" integral transform was introduced and implemented to solve families of FPDEs of the form.

## 2. Basic Definitions and Theorems of" NE" Integral Transform

In this section, we present the definition of "NE" of functions of two variables and the existence conditions and some basic properties of the new double transform are introduced

*2.1. Basic Definitions:*

**Definition 3** . The "NE" integral transform of the continuous function f(x, t) of two variables x > 0 and t > 0 is given by

$$N_2\{f(x,t)\} = E(s;u,v) = \frac{1}{su}\frac{1}{sv}\iint_0^\infty e^{-(\frac{s}{u}x+\frac{s}{v}t)}\, f(x,t)dxdt$$

Clearly, the "NE" is linear, since

$$N_2[\{af(x,t) + bg(x,t)\}] = aN_2\{f(x,t)\} + bN_2\{g(x,t)\}$$

where a and b are constants.
The inverse "NE" integral transform is given by

$$N_2^{-1}[E(s;u,v)] = f(x,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\frac{sx}{u}}\, sds \frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} e^{\frac{st}{v}}\, uvE(s;u,v)du$$

In this article, we implement "NE" integral transform to solve families of FPDEs of the form

$$AD_x^\alpha f(x,t) + BD_t^\beta f(x,t) + CLf(x,t) = z(x,t) \qquad (1)$$
$$x,t > 0, \quad n-1 < \alpha \le n\,, \quad m-1 < \beta \le m \quad and\ m,n\, \epsilon N\,,$$

With the initial conditions

$$\frac{\partial^i f(x,0)}{\partial t^i} = g_i(x) \;\;,\; i = 0,1,\dots,m-1 \qquad (2)$$

and the boundary conditions

$$\frac{\partial^j f(0,t)}{\partial t^j} = h_j(x) \;\;,\; j = 0,1,\dots,n-1 \qquad (3)$$

where A, B and C are real constants, $D_x^\alpha$ and $D_t^\beta$ are the fractional Caputo's derivatives concerning x and t, respectively, L is a linear operator and z(x, t) is the source function. The main motivation of the present study is to expand the applications of "NE" integral transform by using it to solve FPDEs. We show the efficiency of the proposed method by applying the "NE" integral transform to several interesting applications to obtain the exact solutions and analyze the results. The novelty of this work arises from the establishment of a new

simple formula for solving PDEs of fractional orders. The simplicity and applicability of this new formula is illustrated by handling some applications, where we use the new approach to solve some important FPDEs. This article is organized as follows: in the next two sections, we present some basic definitions and theorems related to our work.
A new algorithm for solving families of FPDEs using "NE" integral transform is presented. Several examples are given to demonstrate the proposed technique.

**Definition:** A function f(x,t) defined on [0,X]x[0,T] is called a function of exponential orders $\alpha$ and $\beta$ as $x \to \infty$ and $t \to \infty$ ,if ∃M>0 such that $\forall x > X$ and $\forall t > T$ , we have

$$|f(x,t)| \leq Me^{\alpha x+\beta t}$$

**Theorem 1:** *(Existence condition).* Let f(x,t) be a continuous function on the region $[0,X)x[0,T)$. If f(x, t) is exponential orders λ and γ, then "NE" integral transform of f(x, t) exists, for $Re[s] > 0$, $Re\left[\frac{s}{u}\right] > \alpha$ and $Re\left[\frac{s}{v}\right] > \beta$ .

*Proof:* The "NE" integral transform definition yields that

$$|E(s;u,v)| = \left|\frac{1}{su}\frac{1}{sv}\iint_0^\infty e^{-\left(\frac{s}{u}x+\frac{s}{v}t\right)} f(x,t)dxdt\right| \leq \frac{1}{su}\frac{1}{sv}\iint_0^\infty e^{-\left(\frac{s}{u}x+\frac{s}{v}t\right)} |f(x,t)|dxdt \leq$$

$$\frac{M}{susv}\int_0^\infty e^{-\left(\frac{s}{u}-\alpha\right)x}dx\int_0^\infty e^{-\left(\frac{s}{v}-\beta\right)t}dt = \frac{M}{susv\left(\frac{S}{u}-\alpha\right)\left(\frac{S}{v}-\beta\right)} = \frac{M}{s(s-\alpha u)(s-\beta v)}$$

$Re[s] > 0$, $Re\left[\frac{s}{u}\right] > \alpha$ and $Re\left[\frac{s}{v}\right] > \beta$ .
The proof is completed.

**Theorem 2***: (Derivate Propertie). If* $E(s;u,v) = N_2\{f(x,t)\}$ , then

1. $N_2\left\{\frac{\partial f(x,t)}{\partial t}\right\} = -p_2(s,v)N\{f(x,0)\} + q_2(s,v)N_2\{f(x,t)\} = -\frac{1}{sv}N\{f(x,0)\} + \frac{s}{v}N_2\{f(x,t)\}$
2. $N_2\left\{\frac{\partial f(x,t)}{\partial x}\right\} = -p_1(s,u)N\{f(0,t)\} + q_1(s,u)N_2\{f(x,t)\} = -\frac{1}{su}N\{f(0,t)\} + \frac{s}{u}N_2\{f(x,t)\}$
3. $N_2\left\{\frac{\partial^2 f(x,t)}{\partial x^2}\right\} = -p_1(s,u)[N\{f_x(0,t)\} + q_1(s,u)N\{f(0,t)\}] + q_1(s,u)^2N_2\{f(x,t)\} = -\frac{1}{su}[N\{f_x(0,t)\} + \frac{s}{u}N\{f(0,t)\}] + \frac{s^2}{u^2}N_2\{f(x,t)\}$
4. $N_2\left\{\frac{\partial^2 f(x,t)}{\partial t^2}\right\} = -p_2(s,v)[N\{f_t(x,0)\} + q_2(s,v)N\{f(x,0)\}] + q_2(s,v)^2N_2\{f(x,t)\} = -\frac{1}{sv}[N\{f_x(x,0)\} + \frac{s}{v}N\{f(x,0)\}] + \frac{s^2}{v^2}N_2\{f(x,t)\}$

**Theorem:** *(Convolution theorem)*

Let $N_2[f(x,t)]$ and $N_2[g(x,t)]$ are exists then

$$N_2[(f ** g)] = uvsF(s,u,v)G(s,u,v)$$

Where $f ** g$ is convolution of two function f and g defined by

$$[(f ** g)] = \int_0^x \int_0^t f(x-\rho, t-\tau) g(\rho,\tau) d\rho d\tau$$

$$N_2[(f ** g)] = \frac{1}{su}\frac{1}{sv}\int_0^\infty \int_0^\infty e^{-\left(\frac{s}{u}x+\frac{s}{v}t\right)} (f ** g)(x,t) dxdt$$

$$= \frac{1}{su}\frac{1}{sv}\int_0^\infty \int_0^\infty e^{-\left(\frac{s}{u}x+\frac{s}{v}t\right)} \left[\int_0^x \int_0^t f(x-\rho,t-\tau) g(\rho,\tau)\, d\rho d\tau\right] dxdt$$

Using the Heaviside unit step function, the above equation can be written as

$$N_2[(f ** g)] = \frac{1}{su}\frac{1}{sv}\int_0^\infty \int_0^\infty e^{-\left(\frac{s}{u}x+\frac{s}{v}t\right)} \left[\int_0^\infty \int_0^\infty f(x-\rho,t-\tau) H(x-\rho,t-\tau) g(\rho,\tau)\, d\rho d\tau\right] dxdt$$

$$N_2[(f ** g)] = \int_0^\infty \int_0^\infty g(\rho,\tau) d\rho d\tau \left[\frac{1}{su}\frac{1}{sv}\int_0^\infty \int_0^\infty e^{-\left(\frac{s}{u}x+\frac{s}{v}t\right)} f(x-\rho,t-\tau) H(x-\rho,t-\tau) dxdt\right]$$

$$N_2[(f ** g)] = \int_0^\infty \int_0^\infty g(\rho,\tau) d\rho d\tau \left(e^{-\frac{s}{u}\rho-\frac{s}{v}\tau} N_2[f(x,t)]\right)$$

$$= N_2[f(x,t)] \int_0^\infty \int_0^\infty e^{-\frac{s}{u}\rho-\frac{s}{v}\tau} g(\rho,\tau) d\rho d\tau = uvsN_2[f(x,t)]N_2[g(x,t)]$$

## Algorithm of double "NE" integral transform

In this section, we present the technique of using double "NE" integral transform to solve families of FPDEs. In order to achieve our goal, we have to calculate double "NE" integral transform for the nonlocal Caputo fractional derivative in the following lemma.

*Double "NE" integral transform of Fractional Derivatives*

**Lemma 1:** The double "NE" integral transform for Caputo fractional derivatives can expressed as

1. $N_2[D_x^\alpha f(x,t)] = (\frac{s}{u})^\alpha N_2\{f(x,t)\} - \sum_{i=0}^{n-1} \frac{s^{\alpha-i-2}}{u^{\alpha-i}} \frac{\partial^i f(0,t)}{\partial x^i} = \left(\frac{s}{u}\right)^\alpha E(s;u,v) - \sum_{i=0}^{n-1} \frac{s^{\alpha-i-2}}{u^{\alpha-i}} \frac{\partial^i f(0,t)}{\partial x^i}$,
2. $N_2[D_t^\alpha f(x,t)] = (\frac{s}{v})^\alpha N_2\{f(x,t)\} - \sum_{i=0}^{n-1} \frac{s^{\alpha-i-2}}{v^{\alpha-i}} \frac{\partial^i f(x,0)}{\partial t^i} = \left(\frac{s}{v}\right)^\alpha E(s;u,v) - \sum_{i=0}^{n-1} \frac{s^{\alpha-i-2}}{v^{\alpha-i}} \frac{\partial^i f(x,0)}{\partial t^i}$

$$n-1<\alpha\leq n$$

*Proof of Lemma:* 1. Applying double "NE" integral transform on $D_x^\alpha f(x,t)$ , we obtain:

$$N_2[D_x^\alpha f(x,t)] = N_2\left[\frac{1}{\Gamma(n-\alpha)}\int_0^x (x-\beta)^{n-\alpha-1}\frac{\partial^n f(\beta-t)}{\partial\beta^n}d\beta\right]$$

From the definition of the convolution, we have

$$N_2[D_x^\alpha f(x,t)] = N_2\left[\frac{1}{\Gamma(n-\alpha)}(x^{n-\alpha-1}*\frac{\partial^n f(x,t)}{\partial x^n})\right]=N_t\left[\frac{1}{\Gamma(n-\alpha)}N_x(x^{n-\alpha-1}*\frac{\partial^n f(x,t)}{\partial x^n})\right]$$

Using the convolution property of "NE" transform, we obtain

$$N_2[D_x^\alpha f(x,t)] = N_t\left[\frac{1}{\Gamma(n-\alpha)}(usN_x[x^{n-\alpha-1}]N_x\left[\frac{\partial^n f(x,t)}{\partial x^n}\right])\right]$$

Applying the derivative property of "NE" transform we obtain

$$\begin{aligned}
N_2[D_x^\alpha f(x,t)] &= \frac{1}{\Gamma(n-\alpha)}N_t\left[us\frac{u^{n-\alpha-1}}{s^{n-\alpha+1}}\Gamma(n-\alpha)((\frac{s}{u})^n N_x\{f(x,t)\}-\left(\frac{s}{u}\right)^{n-2}f(0,t)-\cdots\right.\\
&\left.-\frac{1}{su}\frac{\partial^{n-1}f(0,t)}{\partial x^{n-1}})\right]\\
&=\frac{1}{\Gamma(n-\alpha)}N_t\left[\frac{u^{n-\alpha}}{s^{n-\alpha}}\Gamma(n-\alpha)((\frac{s}{u})^n N_x\{f(x,t)\}-\left(\frac{s}{u}\right)^{n-2}f(0,t)-\cdots-\frac{1}{su}\frac{\partial^{n-1}f(0,t)}{\partial x^{n-1}})\right]\\
&=N_t\left[\frac{s^\alpha}{u^\alpha}N_x\{f(x,t)\}-\frac{s^{\alpha-2}}{u^\alpha}f(0,t)-\cdots-\frac{u^{n-\alpha-1}}{s^{n-\alpha+1}}\frac{\partial^{n-1}f(0,t)}{\partial x^{n-1}}\right]\\
&=\left(\frac{s}{u}\right)^\alpha E(s;u,v)-\sum_{i=0}^{n-1}\frac{s^{\alpha-i-2}}{u^{\alpha-i}}\frac{\partial^i f(0,t)}{\partial x^i}
\end{aligned}$$

2.In the same manner can proof the other equal.

## Illustrative Examples

In this section, we apply "NE" transform to obtain solutions of some FPDEs. We consider the initial boundary value problems (1)– (3). To obtain the solution by the new approach, we apply "NE" on both sides of Equation (1), to obtain

$$N_2[AD_x^\alpha f(x,t)]+N_2[BD_t^\beta f(x,t)]+N_2[CLf(x,t)]=N_2[z(x,t)]$$

which implies

$$A\left(\left(\frac{s}{u}\right)^\alpha E(s;u,v)-\sum_{j=0}^{n-1}\frac{s^{\alpha-i-2}}{u^{\alpha-i}}\frac{\partial^i f(0,t)}{\partial x^i}\right)+B\left(\left(\frac{s}{v}\right)^\beta E(s;u,v)-\sum_{i=0}^{m-1}\frac{s^{\beta-j-2}}{v^{\beta-j}}\frac{\partial^j f(x,0)}{\partial t^j}\right)+CN_2[Lf(x,t)]=Z(s;u,v)$$

Furthermore, we apply the single "NE" transform to the ICs (3), and to the BCs (2), to obtain

$N_x[\frac{\partial^j f(x,0)}{\partial t^j}]=N[g_j(x)]=G_j(s)\ \ ,\ j=0,1,\dots,m-1$ (5)

$$N_t[\frac{\partial^i f(0,t)}{\partial t^i}] = N[h_i(x)] = H_i(s) \quad , \ i = 0,1,\dots,n-1 \qquad (6)$$

Simplifying Equation (4), and substituting the values in Equations (5) and (6), we have

$$E(s;u,v) = \frac{1}{A\left(\frac{s}{u}\right)^{\alpha} + B\left(\frac{s}{v}\right)^{\alpha}}\left(A\sum_{i=0}^{n-1}\frac{s^{\alpha-i-2}}{u^{\alpha-i}}H_i(s) + B\sum_{i=0}^{m-1}\frac{s^{\beta-i-2}}{v^{\beta-i}}G_j(s)\right) - CN_2\big[L\big(f(x,t)\big)\big] + Z(s;u,v) \quad (7)$$

Running the inverse "NE" integral transform, on both sides of Equation (7), we obtain

$$f(x,t) = N_2^{-1}\left[\frac{1}{A\left(\frac{s}{u}\right)^{\alpha} + B\left(\frac{s}{v}\right)^{\alpha}}\left(A\sum_{i=0}^{n-1}\frac{s^{\alpha-i-2}}{u^{\alpha-i}}H_i(s) + B\sum_{i=0}^{m-1}\frac{s^{\beta-i-2}}{v^{\beta-i}}G_j(s)\right) - CN_2\big[L\big(f(x,t)\big)\big] + Z(s;u,v)\right]$$

which is the solution of the target problem.

## Conclusions

In this research, "NE" integral transform is applied to the Caputo fractional derivative to obtain a new interesting formula, that is implemented to solve families of FPDEs. We have presented a new method to obtain exact solutions of these equations. We show the reliability and efficiency of the proposed method by presenting some interesting physical applications. In the future, we will pair "NE" integral transform with some iteration methods to solve nonlinear FPDEs, such as nonlinear telegraph equation, nonlinear wave equation, nonlinear Klein– Gordon and nonlinear Fokker–Planck.

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