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Professional paper

STATISTICAL CONVERGES OF SERIES AND STATISTICAL PETTIS INTEGRATION

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Abstract

Pettis and Dunford integrals are the most important concepts concerning the modern theory of probabilities. We extend this to a statistical form. In this paper, we prove the countable additive of the statistical Pettis integral. For this, we need some properties of the unconditional statistical convergence of series in Banach spaces. We give an example where we show that' if a series is statistically unconditionally convergent then it is weakly absolutely statistically convergent.

Keywords: st-convergence, st-measurability st-weakly measurability, st-intergrability st-Dunford integrable, st-Pettis integrable st-uncoditional convergence of series

1. Introduction

It is known that the idea of statistical convergence was given by Zigmund [12]. The concept of statistical convergence was formalized by Steinhaus[11] and Fast [6]. Some years later, the concept was reintroduced by Schoenberg [9]. Statistically, convergence has become an active area of research in recent years. In this presentation, we follow concepts introduced by Fridy[8] about the convergence of sequences and the concept of Schoenberg about integration the basic concept is the statistically Cauchy convergence of Fridy[7].

Let be A_n a subset of ordered natural set N. It said to have density δ(A) if $\delta(A) = \lim_{n\to\infty}\frac{|A_n|}{n}$, where A_n = {k<n : k∈A} and with |A| denotes the cardinality of the set A. It is clear that the finite sets have the density zero and δ(A′)= 1-δ(A) if A'=N-A. If a property P(k)={k : k∈A} holds for all k∈A with δ(A)=1, we say that property P holds for almost all k that is a.a.k. The vectoral sequence x is statistically convergent to the vector(element) L if for each ε >0

$$\lim_{n\to\infty}\frac{1}{n}\left|\{k \le n : \|x_k - L\| \ge \varepsilon\}\right| = 0$$

Or, $\|x_k - L\| < \varepsilon$ a.a.k

Definition 1. A series of elements $x_k \in X$, $k \in N$ of a Banach space X is said to be convergent if the sequence of its partial sums $S_n = \sum_{k=1}^{n} x_k$ statistical convergent in the norm of the space x.

Definition 2. The series $\sum_{k=1}^{\infty} x_k$, $x_k \in$ X, k ∈ N is absolutely statistical convergent if $\sum_K \|x_k\| < \infty$ k∈K∈N,δ(K)= 1.

Proposition 1. If $\sum_{k=1}^{\infty} x_k$ converges statistically absolutely then $\sum_{k=1}^{\infty} x_k$ statistical convergent.

Definition 3. A series $\sum_{k=1}^{\infty} x_k$of elements $x_k \in$ X, k ∈ N of a Banach space X is said to be statistical unconditionally convergent if it statistical convergent for every rearrangement of its terms, i.e. if the series $\sum_K x_{P(n)}$ k∈K∈N ,δ(K)= 1 converges whenever P is a one-to-one mapping of N onto N.

Theorem. For a series $\sum_{k=1}^{\infty} x_k$ of elements $x_k \in$ X, k ∈ N of a Banach space X the following conditions are equivalent:

(a) the series converges statistically unconditionally

(b) all series of the form $x_{n_1} + x_{n_2} + x_{n_3}$ +. . . where $n_1 < n_2 < n_3$ < . . . statistical convergent,

(c) for every bounded sequence $(a)_i$, $(a)_i \in R$ the series $\sum_K \alpha_k x_k$ satatistcal convergent to some element of X.

Proposition 2.

If $\sum_{k=1}^{\infty} x_k$ of elements $x_k \in R$, $k \in N$ is statically unconditionally convergent if and only if Is statistical absolutely convergent.

Definition 4.

A sequence $x_k \in X$,$n \in N$ statistical weakly converges to $x \in X$ if for every $x^* \in X^*$

$\lim_K x^*(x_k) = x^*(x)$

Definition 5.

A series $\sum_{k=1}^{\infty} x_k$ $x_k \in X$, $k \in N$ statistcal weakly convergent to a sum s ∈ X if for every x* ∈ X* the limit $\lim_K x^*(\sum_K x_k) = \lim_K x^*(\sum_K x_k) = x^*(s)$

Theorem. (Orlicz, Pettis)

Let $\sum_{k=1}^{\infty} x_k$ $x_k \in X$, $k \in N$ a series in a Banach space X.

If for each set $A \subset N$ there is $x_A \in X$ such that for each $x^* \in X^*$ we have $\sum_{k\in A} x^*(x_k) = x^*(x_A)$

then the series $\sum_{k=1}^{\infty} x_k$ is unconditionally convergent.

Definition 6:

A series $\sum_{k=1}^{\infty} x_k$ is called statistical weakly absolutely convergent if $\sum_K |x^*(x_k)| <\propto$

Proposition 3.

If $\sum_{k=1}^{\infty} x_k$ is statistically unconditionally convergent then is statistical weakly absolutely convergent.

Example.[10]

For $k \in N$ denote by

$$e_k = \begin{cases} (0, \dots 0, k, 0 \dots) & k\ prime \\ (0, \dots ,0,1,0,0 \dots) & others \end{cases}$$

e_k is the element of c_0 with 1 on the k-th position in the sequence
Since we have

$$\sum_{k=1}^{n} e_k = \begin{cases} (1,2,3,1 \dots n, 0,0 \dots), & k\ prime \\ (1,1,1 \dots\ 1 ,0,0 \dots), & others \end{cases}$$

For $\in N$, we can see immediately that the series $\sum_{k=1}^{\infty} e_k$ does not converge in c_0(in the norm) and the serie $\sum_K e_k$ has the same nature.
Assume that $x^* = {c_0}^* = l_1$, i.e. $x^* = (\alpha_1, \alpha_2, \dots, \alpha_k, \dots)$,$\alpha_k \in R$,$k \in$ N
st-$\sum_{k=1}^{\infty} |\alpha_k| = \|x^*\|_{{c_0}^*} < +\infty$ or $\sum_K |\alpha_k| < +\infty$.
Then $x^*(e_k) = \alpha_k$ for k not prime

$$\sum_K |x^*(e_k)| = \sum_K |\alpha_k| < +\infty$$

i.e the series $\sum_{k=1}^{\infty} e_k$ is stastically weakly convergent .
Since $x^*(\sum_K e_k) = \sum_K \alpha_k$ for $k \in K$ we can see that this sequence converges to $\sum_K \alpha_k = x^*(y)$, where y = (0,0,1,0 ,1,0,1,1..) ,is a sequence which does not belong to c_0. This means that if the series $\sum_{k=1}^{\infty} e_k$ were statistically weakly convergent then its weak sum would be y but c_0 contains such element.

Definition 7. A weakly measurable $f : S \to X$ with x*(f) Lebesgue integrable for every x* ∈ X* is Statistical Pettis integrable if for every measurable E$\subset S$ there is an element x_E ∈ X that satisfies

$$x^*(x_E) = \int_E x^*(f)$$

for every $x^* \epsilon X^*$

Theorem.
If $f : S \to X$ is st- Pettis integrable define for a measurable set $E \subset S$ the function
$\vartheta(E) = Ps - \int_E f\, d\mu \in X$ (the indefinite st-Pettis integral).
The function ϑ is countably additive.
Proof:
Assume that$E_n \subset S$, n∈ N are measurable sets,$E_n \cap E_m = \emptyset$,$n \neq m$.
Then

$$x^*(\vartheta(\bigcup_{n=1}^{\infty} E_n)) = x^*(Ps - \int_{\cup_{n=1}^{\infty} E_n} f\, d\mu = Bs - \int_{\cup_{n=1}^{\infty} E_n} x^*(f)\, d\mu =$$

$$\sum_{n=1}^{\infty}(Bs-\int_{E_n} x^*(f)\,d\mu\,)=\sum_{n=1}^{\infty} x^*(\vartheta(E_n))$$

For every $x^* \in X^*$. This means that υ is weakly countably additive, i.e. the series of real numbers $\sum_{n=1}^{\infty} x^*(\vartheta(E_n))$ is convergent for every $x^* \in X^*$. Hence it is also unconditionally convergent (see Proposition 3.22) and by Theorem 3.21 this means that it is also weakly subseries convergent. The Orlicz-Pettis theorem [10] yields that the series $\sum_{n=1}^{\infty} \vartheta(E_n)$ is unconditionally convergent and henceforth convergent in norm.

While $\sum_{n=1}^{\infty} \vartheta(E_n) = \vartheta(\bigcup_{n=1}^{\infty} E_n)$

The theorem is proved.

Theorem. Let $f: S \to X$ be statistical measurable of the form

$f = g + \sum_K x_n \chi_{E_n}$ $\delta(K) = 1$ (1)

where $g: S \to X$ is measurable and bounded, E_n are pairwise disjoint measurable subsets of S, $x_n \in X$, n ∈ N and E_n, n∈ N can be chosen such that the series $\sum_{n=1}^{\infty} x_n \mu(E_n)$ converges unconditionally in X, and in this case we have

$Ps - \int_E f\,d\mu = Ps - \int_E g\,d\mu + \sum_K x_n\ \mu(E \cap E_n)$ (2)

for every measurable $E \subset S$ and $\delta(K) = 1$

Proof. Assume that f is statistical Pettis integrating of the form (1) . Since g is statistical bounded we have $g \in B_S \subset P_S$ by reposition 3.9 [4] and therefore also

$$h = f - g = \sum_K x_n \chi_{E_n} \in Ps$$

If $E \subset S$ is measurable then, because the indefinite statistical Pettis integral is countably additive by Theorem 3.24 [4], we have

$$\int_E h\,d\mu = \sum_K \int_{E \cap E_n} h\,d\mu = \sum_K x_n \mu(E \cap E_n)\quad, \delta(K) = 1$$

Taking any rearrangement of the series $\sum_K x_n \chi_{E_n}$,, we obtain the same function h, i.e.

$$h = \sum_K x_n \chi_{E_n} = \sum_{k=1}^{\infty} x_{\pi(n)} \chi_{E(\pi(n))}$$

for any one-to-one map $\pi\ of\ K$ onto N and of course

$$\int_E h\,d\mu = \sum_{n=1}^{\infty} \int_{E \cap E_n} h\,d\mu = \sum_{k=1}^{\infty} x_{\pi(n)} \mu(E \cap E_n) \in X$$

Hence the series $\sum_{n=1}^{\infty} x_n \mu(E_n)$ is unconditionally convergent.

To show the converse let us mention that the function $g: S \to X$ being bounded ($\|g(t)\|_X \leq M$ for almost all $t \in S$) is statistical Bochner integrable and therefore $g \in Ps$ by Proposition 3.9(4) Now it suffices to show that $h = \sum_K x_n \chi_{E_n}$, is statistical Pettis integrable provided the series $\sum_{n=1}^{\infty} x_n \, \mu(E_n)$ is unconditionally convergent in X.

Without loss of generality it can be assumed for simplicity that $\mu(E_n) > 0, n \in N$. Assume that $E \subset S$ is measurable.

Then the series

$$\sum_K x_n \, \mu(E \cap E_n) = \sum_K x_n \, \mu(E \cap E_n) \frac{\mu(E \cap E_n)}{\mu(E_n)}$$

is unconditionally convergent in X because $\frac{\mu(E \cap E_n)}{\mu(E_n)} \leq 1$,for all $n \in N$ (Theorem 3.21(4)

If $x^* \in X^*$ then $\sum_K x_n{}^* \, \mu(E \cap E_n)$ converges unconditionally in R and therefore by proposition 3.22 (4) we have

$\int_E |x^*(h)| \, d\mu = \sum_K x_n{}^* \, \mu(E \cap E_n)$ $= x^*(\sum_K x_n \, \mu(E \cap E_n))$

And $h \in Ps$.While

$$Ps - \int_E h \, d\mu = \sum_K x_n \, \mu(E \cap E_n)$$

This yields f = g + h ∈ Ps and also the equality (2)

Theorem 2.3.4. Suppose that X does not contain subspaces isomorphic to C_0 and let $f: S \to X$ be Statistical Dunford integrable. If f is statistical measurable, then f is Statistical Pettis integrable on S.

Proof. Since f is measurable, we have by Proposition 1.1.9 the relation

$$f = g + \sum_K x_n \, \chi_{E_n} \quad \delta(K) = 1$$

where g: S→ X X is measurable and bounded, E_n are pairwise disjoint measurable subsets of S, $x_n \in X$, $n \in N$. Since the interval S is compact, the function $g: S \to X$ is Bochner integrable and by Proposition 2.3.1 also Pettis integrable. The Statistical Dunford integrability $\sum_K x_n \, \chi_{E_n}$ yields the Statistical Dunford integrability (Lebesgue integrability) of $x^*(\sum_K x_n \, \chi_{E_n})$ for every x* ∈ X* and we have also $x^*(\sum_K x_n \, \chi_{E_n})$= $\sum_K x^*(x_n \, \chi_{E_n})$ because the sets En are pairwise disjoint.

Therefore, we have

$\sum_K x^* |x_n \mu(E_n)| < +\infty$ for every x* ∈ X*.

This implies that the series $\sum_K x_n \, \mu(E_n)$ statistikal weakly absolutely converges (see Definition)

Since X does not contain subspaces isomorphic to C_0, by the Bessaga-Pelczyriski Theorem B.22 presented in Appendix B, the series $\sum_K x_n \, \mu(E_n)$ converges unconditionally in X. Hence, $\sum_K x_n \, \chi_{E_n}$ is Pettis integrable by Proposition 2.3.3 and we have f∈P.

Theorem 2.3.5. Suppose that X does not contain subspaces isomorphic to C_0 and let $f: S \to X$ Statistical Dunford integrable. If $Ds - \int_E f \, d\mu \in X$ for every interval J ⊂ S then f is Statistical Pettis integrable on S.

Proof. First of all, we have the following statement

If $J_k \subset S, k \in N$ is a sequence of non-overlapping intervals then $Ds - \int_E f\, d\mu \in X$

Indeed, we have

$$B_S - \int_{\cup J_k} x^*(f) d\mu = \sum_K (B_S - \int_{J_k} x^*(f) d\mu) = \sum_K \left| x^*(Ds - \int_{J_k} f\, d\mu) \right| < +\infty$$

for every x* ∈ X*. Hence the series $\sum_K Ds - \int_{J_k} f\, d\mu$ statistical weakly absolutely converges.

Since X does not contain subspaces isomorphic to C_0 the Bessaga-Pelczyriski Theorem B.22 implies that the series $\sum_K Ds - \int_{J_k} f\, d\mu$ statistically unconditionally converges to a certain element $x_{\cup J_k} \in X$ and $Ds - \int_{\cup J_k} f d\mu = x_{\cup J_k} \in X$ (D) By Theorem (1.11) every open set in R", $m \geq 1$ can be written as a countable union of non-overlapping (closed) intervals an therefore by the statement above we obtain that $Ds - \int_G f d\mu \in X$ for every open set G ⊂ S. If F⊂ S is closed then S \ F is open (in S) and

$$Ds - \int_F f\, d\mu = (Ds - \int_S f d\mu) - (Ds - \int_{S \setminus F} f\, d\mu) \in X$$

Note that if $Z \subset S$ is such that $\mu(Z) = 0$ then $Ds - \int_Z f\, d\mu = 0 \in X$. Let now $E \subset S$ be an arbitrary measurable set. Then by Theorem (3.28) in [WZ77] we have E = H UZ where μ(Z) = 0 and H is of type F_σ i.e where $H = \bigcup_k H_k$, k ∈ N are closed.

Define $L_n = \bigcup_{k=1}^n H_k$ .The sets $H_k \subset S$,k $\in N$ are closed and $L_n \subset L_{n+1}$, $n \in N$. Set $L_0 = \emptyset$, $K_n = L_n / L_{n+1}$, $n \in N$. Then $K_n \cap K_l = \emptyset$ per $n \neq l$ and $H = \bigcup_{n=1}^{\infty} K_n$.

Note that

$$Ds - \int_{k_n} f\, d\mu = Ds - \int_{L_n / L_{n+1}} f\, d\mu = \left(Ds - \int_{L_n} f\, d\mu\,) - (Ds - \int_{L_{n+1}} f\, d\mu \right) \in X$$

Further

$$\int_H x^*(f) d\mu = \int_{\cup\ k_n} x^*(f)\, d\mu = \sum_n \int_{k_n} x^*(f) d\mu = \sum_n x^* \left(Ds - \int_{k_n} f\, d\mu \right) d\mu$$

And

$$\sum_n \left| x^* \left(Ds - \int_{k_n} f\, d\mu \right) d\mu \right| = \sum_n \left| Bs - \int_{k_n} x^*(f) d\mu \right| \leq$$

$\sum_n Bs - \int_{k_n} |x^*(f)|\, d\mu = \int_{\cup K_n} |x^*(f)|\, d\mu = \int_H |x^*(f) d\mu| < +\infty$

for every x* ∈ X*.

Similarly as above the Bessaga-Pelczynski Theorem B.22 from Appendix B implies that the series $Ds - (\int_{k_n} f\, d\mu)$ statistical unconditionally converges to a certain element $x_H \in X$ and

$$Ds-(\int_{H} f\, d\mu)=x_H \in X$$

Hense

$$Ds-\int_{E} fd\mu = Ds-\int_{H} fd\mu + Ds-\int_{Z} fd\mu \in X$$

Theorem:

We have $Bs \subset Ps$ and the inclusion is proper for general Banach spaces X, i.e., there exist Banach spaces X and functions f: S→ X which are Statistical Pettis integrable but not Statistical Bochner integrable.

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