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GENERALIZED INVERSES, LIMITS, AND PARTITIONED MATRICES

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Abstract

Our final result is interconnectedness between the generalized inverses of partitioned matrices and limit representations of generalized inverses. These results are established using relationships between generalized Schur complements $(A/R)_g$ and $(A/T)_g$ of an appropriate partitioned matrix $A=\begin{bmatrix} R & -ST \\ TU & T \end{bmatrix}$. Also, some essential relations are investigated between the blocks involved in generalized inverses of A and generalized inverses of the Schur complements $(A/R)_g$ and $(A/T)_g$. Some rank equalities on generalized inverses are obtained.
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1. Introduction

This paper aims to establish relationship between the generalized Schur complements $(A/R)_g$ and $(A/T)_g$ of a partitioned matrix $A=\begin{bmatrix} R & -ST \\ TU & T \end{bmatrix}$ satisfying $\mathcal{R}(TU)\subset\mathcal{R}(R)$ and $\mathcal{C}(ST)\subset\mathcal{C}(R)$. Also, we derive a few relations between the blocks involved in generalized inverses of A and generalized inverses of the Schur complements $(A/R)_g$ and $(A/T)_g$. Moreover, we obtain a few additional results related to $\operatorname{rank}\left(\alpha\mathbf{I}+Y^*Z\right)$ and correlations between the limit $\lim_{\alpha\to 0}\left(\alpha\mathbf{I}+Y^*Z\right)^{-1}Y^*$ and g-inverses of the partitioned matrix $\begin{bmatrix} \alpha\mathbf{I}_q & -Y^* \\ Z & \mathbf{I}_p \end{bmatrix}$.

To ensure completeness in our presentation, we list some basic labels and notions. The standard notation $\mathbb{C}^{m\times n}$ stands for $m\times n$ matrices over complex numbers $\mathbb{C}$ and $\mathbb{C}_r^{m\times n}=\left\{X\in\mathbb{C}^{m\times n}\,\middle|\,\operatorname{rank}(X)=r\right\}$. An appropriate zero matrix is denoted by $\mathbf{O}$, while $\mathbf{I}_p$ denotes the $p\times p$ identity matrix. For any matrix A, the row and column space of A are denoted by $\mathcal{R}(A)$ and $\mathcal{C}(A)$, respectively.

For a given $A\in\mathbb{C}^{m\times n}$ the following equations in X are important in classifying generalized inverses:

$$(1)\ \ AXA = A, \qquad (2)\ \ XAX = X, \qquad (3)\ \ (AX)^* = AX, \qquad (4)\ \ (XA)^* = XA.$$

Important equations in the case $m = n$ are

$$(5)\ \ AX = XA, \qquad (1^k)\ \ A^{k+1}X = A^k,$$

where *k* denotes the index of A, defined by $\operatorname{ind}(A) = \min\left\{k \,\middle|\, \operatorname{rank}(A^{k+1}) = \operatorname{rank}(A^k)\right\}$.

The set of generalized inverses obeying the equations represented in S is denoted by $A\{\mathrm{S}\}$. A matrix from $A\{\mathrm{S}\}$ is called an S-inverse of A and denoted by $A^{(\mathrm{S})}$. A solution to the equation (1) is said to be a *g*-inverse, or $\{1\}$-inverse (or inner inverse) of A. If *X* satisfies (1) and (2), it is known as a reflexive *g*-inverse of A, whereas the Moore-Penrose inverse $X = A^\dagger$ fulfills all the equations (1)–(4). A matrix A^D is said to be the Drazin inverse of A if (1^k), (2), (5) are satisfied. The group inverse $A^\#$ represents the unique element $A\{1, 2, 5\}$ and exists under the restriction $\operatorname{ind}(A) = 1$. A matrix $X \in A\{\mathrm{S}\}$ which fulfills $\mathrm{R}(X) = \mathrm{R}(E)$ and $\mathrm{N}(X) = N(F)$ will be termed as $A^{(S)}_{\mathrm{R}(E),\mathrm{N}(F)}$. An arbitrary matrix contained in $A\{2\}$ is known as an outer generalized inverse of A.

The Schur complement of E in a block matrix $A = \begin{bmatrix} E & F \\ G & H \end{bmatrix}$ is defined as

$$(A/E) = H - GE^{-1}F \ .$$

Various generalizations of Schur complement are introduced utilizing miscellaneous generalized inverses of E. Consequently, heterogeneous representations and characterizations of generalized inverses of block matrices are introduced in terms of generalized Schur complements. The Schur complement and its extensions have been exploited extensively in the matrix theory, statistics, calculation of large-scale generalized inverses, and numerical analysis [1, 4, 5, 7, 8, 9, 10, 13, 16, 24, 28, 30]. Generalized Schur complement based on the Drazin inverse $(A/E)_D = H - GE^D F$ and initiated representations of A^D were investigated in [18, 25]. Generalized Shur complement based on the group inverse $(A/E)_\# = H - GE^\# F$ and initiated representations of $A^\#$ were investigated in [23]. Generalization of the Shur complement based on the Moore-Penrose inverse $(A/E)_\dagger = H - GE^\dagger F$ and initiated representations of $A^\dagger$ were investigated in [12, 15]. Carlson in [11] introduced the Schur complement in terms of *g*-inverses. The generalized Schur complement of E and H in $A = \begin{bmatrix} E & F \\ G & H \end{bmatrix}$, respectively, in terms of *g*-inverses are defined by the expressions [10, 11, 30]

$$(A/E)_g = H - GE^{(1)}F, \qquad (A/H)_g = E - FH^{(1)}G.$$

The generalized Schur complement based on the generalized inverse $A^{(2)}_{\mathrm{R}(E),\mathrm{N}(F)}$ was proposed and investigated in [38]:

$$(A/E)_{(2)} = H - GE^{(2)}_{\mathrm{R}(E),\mathrm{N}(F)}F, \qquad (A/H)_{(2)} = E - FH^{(2)}_{\mathrm{R}(E),\mathrm{N}(F)}G.$$

The most general structure of our research is organized according to the following scheme. Preliminaries, motivation, and description of the main results are described in Section 2. Section 3 derives the main results of the research. Concluding remarks are given in Section 4.

2. Preliminaries and description of main results

The results stated in this section are available in literature, and are restated here for the sake of completeness.

Theorem 2.1. [19] Let E be an $r\times t$ matrix, F is an $r\times u$ matrix, V be an $s\times t$ matrix, and W be an $s\times u$ matrix. Assume $\mathcal{R}(V)\subset\mathcal{R}(E)$ and $\mathcal{C}(F)\subset\mathcal{C}(E)$. Then, for any g-inverse $G=\begin{bmatrix} G_{11} & G_{12}\\ G_{21} & G_{22}\end{bmatrix}$ of the partitioned matrix $A=\begin{bmatrix} W & V\\ F & E\end{bmatrix}$, the $u\times s$ submatrix G_{11} is a g-inverse of $W-VE^{(1)}F=\left(A/E\right)_g$. Similarly, for any g-inverse $H=\begin{bmatrix} H_{11} & H_{12}\\ H_{21} & H_{22}\end{bmatrix}$ of the partitioned matrix $B=\begin{bmatrix} E & F\\ V & W\end{bmatrix}$ the $u\times s$ submatrix H_{22} is a g-inverse of $W-VE^{(1)}F=\left(B/E\right)_g$. Further,

$$\operatorname{rank}\left(\begin{bmatrix} W & V\\ F & E\end{bmatrix}\right)=\operatorname{rank}\left(\begin{bmatrix} F & E\\ W & V\end{bmatrix}\right)=\operatorname{rank}\left(E\right)+\operatorname{rank}\left(W-VE^{(1)}F\right)$$
$$=\operatorname{rank}\left(E\right)+\operatorname{rank}\left(B/E\right)_g=\operatorname{rank}\left(E\right)+\operatorname{rank}\left(A/E\right)_g.$$

Corollary 2.1. [19] For any g-inverse $G=\begin{bmatrix} G_{11} & G_{12}\\ G_{21} & G_{22}\end{bmatrix}$ of the matrix $A=\begin{bmatrix} R & -ST\\ TU & T\end{bmatrix}$, the block G_{11} is a g-inverse of $R+STU=\left(A/T\right)_g$. Further,

$$\operatorname{rank}\left(R+STU\right)=\operatorname{rank}\left(A\right)-\operatorname{rank}\left(T\right).$$

Theorem 2.2. [19] The matrix

$$R^{(1)}-R^{(1)}ST\left(T+TUR^{(1)}ST\right)^{(1)}TUR^{(1)}$$

is a g-inverse of $R+STU$ if $\mathcal{R}(TU)\subset\mathcal{R}(R)$ and $\mathcal{C}(ST)\subset\mathcal{C}(R)$.

Lemma 2.1. [2] Let A represents an $r\times s$ matrix. For arbitrary $r\times t$ matrix B, $\mathcal{C}(B)\subset\mathcal{C}(A)$ Û $B=AA^{(1)}B$. Also, for any $u\times s$ matrix C, $\mathcal{R}(C)\subset\mathcal{R}(A)$ Û $C=CA^{(1)}A$.

Theorem 2.3. [19] Let $Q=T+TUR^{(1)}ST$. If $\mathcal{R}(TU)\subset\mathcal{R}(R)$ and $\mathcal{C}(ST)\subset\mathcal{C}(R)$ then

$$\operatorname{rank}\left(R+STU\right)=\operatorname{rank}\left(R\right)-\left[\operatorname{rank}\left(T\right)-\operatorname{rank}\left(Q\right)\right].$$

The following main results of this research are emphasized.

- We investigate a relation between the generalized Schur complement $(A/R)_g$ and the generalized Schur complement $(A/T)_g$ of a partitioned matrix

$$A = \begin{bmatrix} R & -ST \\ TU & T \end{bmatrix}$$

  whose blocks satisfy the conditions $\mathcal{R}(TU) \subset \mathcal{R}(R)$ and $\mathcal{C}(ST) \subset \mathcal{C}(R)$. Also, we introduce a few identities between the generalized Schur complements $(A/R)_g$ and $(A/T)_g$ and blocks of an arbitrary {1}-inverse of A.
- In a partial case, for a partitioned matrix of the form

$$B = \begin{bmatrix} \alpha \mathbf{I}_q & -Y^* \\ Z & \mathbf{I}_p \end{bmatrix}$$

  we get several identities between blocks contained in {1}-inverses of B and the limit expression $L = \lim_{\alpha \to 0} \left(\alpha \mathbf{I}_q + Y^* Z\right)^{(1)} Y^*, \alpha > 0$. Such a principle leads to a generalization of the known result [33, Lemma 2.1].

- These results can be applied in the computation of the limit L in the case of its existence. Then various classes of generalized inverses of a given $p \times q$ matrix B, which can be expressed by the limit L, are expressed in terms of blocks contained in {1}-inverses of B. We discuss all these classes in details. As far as we know, our results are the first correlations between the generalized inverses of partitioned matrices and the limit representation of generalized inverses.

- Moreover, we obtain a few additional results about the rank of the matrix $\alpha \mathbf{I}_q + Y^* Z,\ \alpha > 0$.

3. Results

In the following theorem, we introduce several identities between the generalized Schur complements of a particular partitioned matrix and blocks contained in its {1}-inverse.

Theorem 3.1. Let R, S, T and U be matrices of the order $n \times q$, $n \times m$, $m \times p$ and $p \times q$, respectively. Consider the partitioned matrix

$$A = \begin{bmatrix} R & -ST \\ TU & T \end{bmatrix}$$

which satisfies the conditions

$$\mathcal{R}(TU) \subset \mathcal{R}(R) \text{ and } \mathcal{C}(ST) \subset \mathcal{C}(R) \tag{3.1}$$

Then the generalized Schur complements of T and R in A satisfy

$$(A/T)_g^{(1)} = R^{(1)} - R^{(1)} ST (A/R)_g^{(1)} TUR^{(1)} \tag{3.2}$$

for some {1}-inverses of the generalized Schur complements $(A/T)_g$ and $(A/R)_g$ and {1}-inverse $R^{(1)}$ of R.

Also, if

$$G=\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$$

is a g-inverse of A, the following additional results are obtained:

$$\begin{aligned} G_{11} &= R^{(1)} - R^{(1)}ST\left(A/R\right)_g^{(1)} TUR^{(1)} \\ \left(A/T\right)^{(1)} &= R^{(1)} - R^{(1)}STG_{22}TUR^{(1)} \\ G_{11} &= R^{(1)} - R^{(1)}STG_{22}TUR^{(1)} \end{aligned} \tag{3.3}$$

Proof. The generalized Schur complement of T in A is equal to

$$\left(A/T\right) = R - \left(-ST\right)T^{(1)}TU = R + STU \tag{3.4}$$

Using (3.1) and the result from Theorem 2.2, it is concluded

$$R^{(1)} - R^{(1)}ST\left(T + TUR^{(1)}ST\right)^{(1)} TUR^{(1)} \in \left(R + STU\right)\{1\} \tag{3.5}$$

Also, using

$$\left(A/R\right)_g = T - TUR^{(1)}\left(-ST\right) = T + TUR^{(1)}ST \tag{3.6}$$

in conjunction with (3.4),(3.5),(3.6), it is concluded

$$\begin{aligned} \left(A/T\right)^{(1)} &= \left(R + STU\right)^{(1)} = R^{(1)} - R^{(1)}ST\left(T + TUR^{(1)}ST\right)^{(1)} TUR^{(1)} \\ &= R^{(1)} - R^{(1)}ST\left(A/R\right)_g^{(1)} TUR^{(1)} \end{aligned}$$

In this way, the equality (3.2) is verified.

According to Corollary 1.1, the block G_{11} is a generalized inverse of the generalized Schur complement $\left(A/T\right)_g$:

$$G_{11} = \left(A/T\right)_g^{(1)} \tag{3.7}$$

On the other hand, using (3.1) and applying the second part of Theorem 2.1, we obtain

$$G_{22} = \left(A/R\right)_g^{(1)} \tag{3.8}$$

Equalities in (3.3) follow from (3.2),(3.7) and (3.8). □

Particularly, when the partitioned matrix of the form $B=\begin{bmatrix} \alpha\mathbf{I}_q & -Y^* \\ Z & \mathbf{I}_p \end{bmatrix}$ is considered, the limiting expressions $\lim_{\alpha\to 0}\left(\alpha\mathbf{I}_q + Y^*Z\right)^{(1)} Y^*$ can be expressed in terms of limit expressions involving blocks of {1}-inverses of B.

Theorem 3.2. Let Y, Z be arbitrary matrices of the order $p\times q$. Then the following identities are valid:

$$\begin{aligned} &\lim_{\alpha\to 0}\left(\alpha\mathbf{I}_q+Y^*Z\right)^{(1)}Y^*=\lim_{\alpha\to 0}C_{11}Y^*;\\ &\lim_{\alpha\to 0}Y^*\left(\alpha\mathbf{I}_p+ZY^*\right)^{(1)}=\lim_{\alpha\to 0}\frac{1}{\alpha}Y^*C_{22}; \end{aligned} \tag{3.9}$$

$$\begin{aligned} &\lim_{\alpha\to 0}Z\left(\alpha\mathbf{I}_q+Y^*Z\right)^{(1)}=\lim_{\alpha\to 0}ZC_{11};\\ &\lim_{\alpha\to 0}\left(\alpha\mathbf{I}_p+ZY^*\right)^{(1)}Z=\lim_{\alpha\to 0}\frac{1}{\alpha}C_{22}Z, \end{aligned} \tag{3.10}$$

where $C=\begin{bmatrix}C_{11} & C_{12}\\ C_{21} & C_{22}\end{bmatrix}$ is an arbitrary g-inverse of the matrix $B=\begin{bmatrix}\alpha\mathbf{I}_q & -Y^*\\ Z & \mathbf{I}_p\end{bmatrix}$.

If the matrix B is invertible, it follows

$$\begin{aligned} \lim_{\alpha\to 0}\left(\alpha\mathbf{I}_q+Y^*Z\right)^{(1)}Y^* &=\lim_{\alpha\to 0}C_{11}Y^*\\ &=\lim_{\alpha\to 0}Y^*\left(\alpha\mathbf{I}_p+ZY^*\right)^{(1)}=\lim_{\alpha\to 0}\frac{1}{\alpha}Y^*C_{22}\\ &=\lim_{\alpha\to 0}C_{22}; \end{aligned} \tag{3.11}$$

$$\begin{aligned} \lim_{\alpha\to 0}Z\left(\alpha\mathbf{I}_q+Y^*Z\right)^{(1)} &=\lim_{\alpha\to 0}ZC_{11}\\ &=\lim_{\alpha\to 0}\left(\alpha\mathbf{I}_p+ZY^*\right)^{(1)}Z=\lim_{\alpha\to 0}\frac{1}{\alpha}C_{22}Z\\ &=-\lim_{\alpha\to 0}C_{21}. \end{aligned} \tag{3.12}$$

Proof. Starting from

$$Y^*=Y^*\left(\mathbf{I}_p\right)^{(1)}\mathbf{I}_p,\qquad Z=\mathbf{I}_p\left(\mathbf{I}_p\right)^{(1)}Z,$$

an application of Lemma 2.1 enables the following inclusions:

$$\mathrm{R}\,(Y^*)\subset\mathrm{R}\,(\mathbf{I}_p)\ \text{ and }\ \mathrm{C}(Z)\subset\mathrm{C}(\mathbf{I}_p).$$

According to Theorem 2.1, C_{11} is a g-inverse of the matrix

$$\left(B/\mathbf{I}_p\right)=\alpha\mathbf{I}_p-\left(-Y^*\right)\left(\mathbf{I}_p\right)^{(1)}Z=\alpha\mathbf{I}_q+Y^*Z.$$

Hence, we obtain

$$\lim_{\alpha\to 0}C_{11}Y^*=\lim_{\alpha\to 0}\left(\alpha\mathbf{I}_q+Y^*Z\right)^{(1)}Y^*. \tag{3.13}$$

On the other hand, using

$$Z=Z\left(\alpha\mathbf{I}_q\right)^{(1)}\left(\alpha\mathbf{I}_q\right),\qquad Y^*=\left(\alpha\mathbf{I}_q\right)\left(\alpha\mathbf{I}_q\right)^{(1)}Y^*,$$

it is obtained

$$\mathrm{R}(Z) \subset \mathrm{R}(\alpha \mathbf{I}_q) \text{ and } \mathrm{C}(Y^*) \subset \mathrm{C}(\alpha \mathbf{I}_q).$$

In view of Theorem 2.1, C_{22} is a g-inverse of the matrix

$$\left(B/(\alpha \mathbf{I}_q)\right)_g = \mathbf{I}_p - Z(\alpha \mathbf{I}_q)^{(1)}\left(-Y^*\right) = \frac{1}{\alpha}\left(\alpha \mathbf{I}_p + ZY^*\right)$$

This implies

$$\lim_{\alpha \to 0} \frac{1}{\alpha} Y^* C_{22} = \lim_{\alpha \to 0} Y^* \left(\alpha \mathbf{I}_p + ZY^*\right)^{(1)}. \tag{3.14}$$

In a similar way one can verify the following:

$$\lim_{\alpha \to 0} ZC_{11} = \lim_{\alpha \to 0} Z\left(\alpha \mathbf{I}_q + Y^* Z\right)^{(1)}; \tag{3.15}$$

$$\lim_{\alpha \to 0} \frac{1}{\alpha} C_{22} Z = \lim_{\alpha \to 0} \left(\alpha \mathbf{I}_p + ZY^*\right)^{(1)} Z. \tag{3.16}$$

If the matrix B is invertible, we have $BC = CB = \mathbf{I}$. Using the equation $BC = I$, it is not difficult to verify

$$C_{12} = \frac{1}{\alpha} Y^* C_{22} = C_{11} Y^*. \tag{3.17}$$

The part (3.11) of the proof follows from (3.13), (3.14) and (3.17).

Also, using $CB = I$, one can verify the following

$$C_{21} = -\frac{1}{\alpha} C_{22} Z = -ZC_{11}. \tag{3.18}$$

The part (3.12) of the proof is implied by (3.15), (3.16) and (3.18). □

Remark 3.1. Results (3.11) and (3.12) of Theorem 3.2 are a generalization of the known result introduced in [33, Lemma 2.1].

We now assume the existence of the general limit representation $L = \lim_{\alpha \to 0}\left(\alpha \mathbf{I}_q + Y^* Z\right)^{-1} Y^*$ which is investigated in [33]. As a consequence of the previous results, we obtain representations of various classes of generalized inverses in terms of the limits of expressions involving blocks of {1}-inverses of the matrix B.

Theorem 3.3. Consider $A \in \mathbb{C}_r^{m \times n}$, two $p \times q$ matrices Y and Z and the block matrix $B = \begin{bmatrix} \alpha \mathbf{I}_q & -Y^* \\ Z & \mathbf{I}_p \end{bmatrix}$. Let $C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$ be an arbitrary {1}-inverse of B. Then the subsequent statements are true:

(i) $Y = Z = A \;\Rightarrow\; \lim_{\alpha\to 0}\frac{1}{\alpha}Y^*C_{22} = \lim_{\alpha\to 0}C_{11}Y^* = A^\dagger.$

(ii) $Y^* = A^k,\quad Z = A,\; k \ge \mathrm{ind}(A) \;\Rightarrow\; A^D = \lim_{\alpha\to 0}C_{11}Y^* = \lim_{\alpha\to 0}\frac{1}{\alpha}Y^*C_{22}.$

(iii) $Y^*Z = A,\quad Y^* = A^k,\; k = \mathrm{ind}(A) \;\Rightarrow\; A^D = \lim_{\alpha\to 0}C_{11}^{k+1}Y^*.$

(iv) $Y^* = A^k,\quad k = \mathrm{ind}(A),\quad Z = \mathbf{I}_n \Rightarrow\; AA^D = \lim_{\alpha\to 0}\frac{1}{\alpha}A^kC_{22} = \lim_{\alpha\to 0}C_{11}A^k.$

Proof. (i) Since the matrix A^*A is positive semidefinite and $\alpha > 0$, the matrix $\alpha\mathbf{I} + A^*A$ is positive definite. Using invertibility of the matrix $\alpha\mathbf{I} + A^*A$, according to Theorem 3.2 we get

$$\lim_{\alpha\to 0}C_{11}A^* = \lim_{\alpha\to 0}\left(\alpha\mathbf{I} + A^*A\right)^{-1}A^* = \lim_{\alpha\to 0}A^*\left(\alpha\mathbf{I} + AA^*\right)^{-1} = \lim_{\alpha\to 0}\frac{1}{\alpha}A^*C_{22}.$$

Now, it is sufficient to use the limit representation of the Moore-Penrose inverse from [2, 3].

(ii) Application of the known result from [26, 31] implies the existence of the limit expression

$$\lim_{\alpha\to 0}A^*\left(\alpha\mathbf{I}_n + A^{k+1}\right)^{-1}A^k, \quad k = \mathrm{ind}(A).$$

Using the limit representation of the Drazin inverse from [26, 31] and Theorem 3.2, we get

$$A^D = \lim_{\alpha\to 0}\left(\alpha\mathbf{I}_n + A^{k+1}\right)^{-1}A^k = \lim_{\alpha\to 0}\left(\alpha\mathbf{I}_n + A^kA\right)^{-1}A^k$$
$$= \lim_{\alpha\to 0}C_{11}Y^* = \lim_{\alpha\to 0}A^k\left(\alpha\mathbf{I}_n + AA^k\right)^{-1} = \lim_{\alpha\to 0}\frac{1}{\alpha}Y^*C_{22}.$$

where $Y^* = A^k,\;\; Z = A$.

(iii) Follows from the following result, introduced in [20]:

$$A^D = \lim_{\alpha\to 0}\left(\alpha\mathbf{I}_n + A\right)^{-(k+1)}A^k.$$

(iv) Follows from the following statement, introduced in [35, 36]: If $\mathrm{ind}(A) = k$, then

$$\lim_{\alpha\to 0}\left(\alpha\mathbf{I}_n + A^k\right)^{-1}A^k = AA^D.$$

The proof is complete. □

In the following theorem we obtain a representation of generalized inverse $A_{T,S}^{(2)}$ in terms of limiting expressions involving blocks contained in g-inverses of a partitioned matrix.

Theorem 3.4. Let $A \in \mathbb{C}_r^{m\times n}$, let T be a subspace of $\mathbb{C}^n$ of dimension $s \le r$, and S be a subspace of $\mathbb{C}^m$ of dimension $m - s$. Further, assume that $G \in \mathbb{C}^{n\times m}$ satisfies $\mathcal{R}(G) = T,\; \mathcal{N}(G) = S$. If $A_{T,S}^{(2)}$ exists, it follows

$$A_{T,S}^{(2)} = \lim_{\alpha\to 0}C_{11}G = \lim_{\alpha\to 0}\frac{1}{\alpha}GC_{22},$$

where $C=\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$ is any {1}-inverse of

$$B=\begin{bmatrix} \alpha\mathbf{I}_n & -G \\ A & \mathbf{I}_m \end{bmatrix}.$$

When B is of full rank it follows additionally

$$A_{T,S}^{(2)}=\lim_{\alpha\to 0} C_{12}=-\lim_{\alpha\to 0} C_{21}.$$

Proof. In this case, identities in (3.9) and (3.10) can be written in the form

$$\begin{aligned} \lim_{\alpha\to 0} C_{11}G &= \lim_{\alpha\to 0}\left(\alpha\mathbf{I}_n+GA\right)^{(1)}G \\ \lim_{\alpha\to 0}\frac{1}{\alpha}GC_{22} &= \lim_{\alpha\to 0} G\left(\alpha\mathbf{I}_n+AG\right)^{(1)}. \end{aligned} \tag{3.19}$$

The following limit representation of the generalized inverse $A_{T,S}^{(2)}$, from [35, 36, 37]

$$A_{T,S}^{(2)}=\lim_{\alpha\to 0}\left(\alpha\mathbf{I}_n+GA\right)^{-1}G,$$

deduces invertibility of $\alpha\mathbf{I}_n+GA$ in the case when $A_{T,S}^{(2)}$ exists. The proof can be completed using (3.19) and

$$\lim_{\alpha\to 0}\left(\alpha\mathbf{I}_n+GA\right)^{-1}G=\lim_{\alpha\to 0}G\left(\alpha\mathbf{I}_n+AG\right)^{-1}.$$

For a full rank matrix B the proof follows from (3.11) and (3.12). □

In a similar way as in [33, Theorem 2.3], one can verify the following statements.

Corollary 3.1. Let $C=\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$ be an arbitrary g-inverse of the partitioned matrix $B=\begin{bmatrix} \alpha\mathbf{I}_q & -Y^* \\ Z & \mathbf{I}_p \end{bmatrix}$, where Y and Z are $p\times q$ matrices. Consider $A\in \mathbb{C}_r^{m\times n}$, two arbitrary matrices $W_1\in\mathbb{C}^{n\times r}$ and $W_2\in\mathbb{C}^{r\times m}$ which satisfy $\operatorname{rank}\left(W_2AW_1\right)=r$ and two arbitrary matrices $G\in\mathbb{C}^{n\times s}$ and $H\in\mathbb{C}^{s\times m}$ satisfying $\operatorname{rank}\left(HAG\right)=s\le r$. Then the subsequent statements are equivalent:

(i) $D\in A\{1,2\} \quad\Leftrightarrow\quad D=\lim_{\alpha\to 0}\frac{1}{\alpha}Y^*C_{22}=\lim_{\alpha\to 0}C_{11}Y^*, \quad Z=A, \quad Y^*=W_1W_2;$

(ii) $D\in A\{1,2\} \quad\Leftrightarrow\quad D=\lim_{\alpha\to 0}\frac{1}{\alpha}Y^*C_{22}=\lim_{\alpha\to 0}C_{11}Y^*, \quad Y=Z=W_2AW_1;$

(iii) $D\in A\{2\} \quad\Leftrightarrow\quad D=\lim_{\alpha\to 0}\frac{1}{\alpha}Y^*C_{22}=\lim_{\alpha\to 0}C_{11}Y^*, \quad Z=A, \quad Y^*=GH;$

(iv) $X\in A\{2\} \quad\Leftrightarrow\quad X=\lim_{\alpha\to 0}\frac{1}{\alpha}Y^*C_{22}=\lim_{\alpha\to 0}C_{11}Y^*, \quad Y=Z=HAG.$

Remark 3.2. A generalization of the Leverrier-Feddeev algorithm for the implementation of the limit representation $\lim_{\alpha\to 0}\left(\alpha\mathbf{I}_q+Y^*Z\right)^{-l}Y^*$ was introduced in [33]. In this way, a general form of various modifications of the Leverrier-Feddeev algorithm for calculating generalized inverses was obtained in [33]. Covered modifications were investigated in [14, 17, 21, 34]. Also, a method for computing generalized inverses, which is based on the system of differential equations, and arising from the corresponding limit representations, was introduced in [21] and [34].

In this paper we develop an additional method to calculate $\lim_{\alpha\to 0}\left(\alpha\mathbf{I}_q+Y^*Z\right)^{-l}Y^*$, based on the application of blocks contained in g-inverses of the partitioned matrix $\begin{bmatrix}\alpha\mathbf{I}_q & -Y^*\\ Z & \mathbf{I}_p\end{bmatrix}$. Because of (3.9), limit expressions

$$\lim_{\alpha\to 0}C_{11}Y^*=\lim_{\alpha\to 0}\frac{1}{\alpha}Y^*C_{22}$$

are applicable even in the case when the inverse $\left(\alpha\mathbf{I}_q+Y^*Z\right)^{-1}$ cannot be calculated by a computer.

Theorem 3.5. Let $C=\begin{bmatrix}C_{11} & C_{12}\\ C_{21} & C_{22}\end{bmatrix}$ be an arbitrary g-inverse of the partitioned matrix $B=\begin{bmatrix}\alpha\mathbf{I}_q & -Y^*\\ Z & \mathbf{I}_p\end{bmatrix}$, where Y and Z are $p\times q$ matrices . If $\alpha>0$ is a given real number, we get the following identities:

$$\operatorname{rank}\left(\alpha\mathbf{I}_q+Y^*Z\right)=\operatorname{rank}\left(B\right)-p=q-p+\operatorname{rank}\left(\alpha\mathbf{I}_p+ZY^*\right) \qquad (3.20)$$

Proof. The first equality in (3.20) follows from Corollary 2.1. Also, using

$$\mathrm{R}\,(Z)\subset\mathrm{R}\,(\alpha\mathbf{I}_q),\qquad \mathrm{C}(Y^*)\subset\mathrm{C}(\alpha\mathbf{I}_q),$$

and applying the results of Theorem 2.3 to the matrix B, we get

$$\begin{aligned}\operatorname{rank}\left(\alpha\mathbf{I}_q+Y^*Z\right)&=q-\left[p-\operatorname{rank}\left(\mathbf{I}_p+Z\frac{1}{\alpha}\mathbf{I}_qY^*\right)\right]\\&=q-p+\operatorname{rank}\left(\alpha\mathbf{I}_p+ZY^*\right).\end{aligned}$$

which finalizes the proof. □

4. Conclusion

Relationship between the generalized Schur complements $(A/R)_g$ and $(A/T)_g$ of the partitioned matrix $A=\begin{bmatrix}R & -ST\\ TU & T\end{bmatrix}$ are considered under the restrictions $\mathrm{R}\,(TU)\subset\mathrm{R}\,(R)$ and $\mathrm{C}(ST)\subset\mathrm{C}(R)$. In addition, we propose several relations between the blocks involved in generalized inverses of A and generalized inverses of the Schur complements $(A/R)_g$ and $(A/T)_g$. Moreover, we obtain a few additional results related to $\operatorname{rank}\left(\alpha\mathbf{I} + Y^*Z\right)$ and correlations between $\lim_{\alpha\to 0}\left(\alpha\mathbf{I} + Y^*Z\right)^{-l} Y^*$ and g-inverses of $\begin{bmatrix} \alpha\mathbf{I}_q & -Y^* \\ Z & \mathbf{I}_p \end{bmatrix}$. Some rank equalities on generalized inverses are obtained.

Actual results are derived using the generalized Schur complement based on {1}-inverses. Further research may refer to analogous results based on generalized Schur complement resulting from different generalized inversions, such as the Moore-Penrose inverse or outer generalized inverses.

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