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CHARACTERIZATION AND APPLICABILITY OF SOME SPECIAL TYPES OF MOORE GRAPHS

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Abstract

Various types of complex networks, such as telecommunication networks, are modeled using graphs whose vertices are telecommunication stations and whose edges are cables. During the construction of such networks, there are various restrictions on the vertices or edges. The condition is usually set on the highest degree of the vertex of that graph (that network) and the diameter of the graph (in that network). When studying such graphs, two typical problems arise: (i) degree/diameter problem: for given natural numbers λ and L, it is necessary to find the largest possible number of vertices $n_{\lambda,L}$ of a connected graph whose largest degree is equal to λ, and whose diameter is less than or equal to L; (ii) degree/girth problem: with the given natural numbers $d \geq 2$ and $g \geq 3$, the smallest possible number of vertices of a connected regular graph whose degree of regularity is equal to d and girth is g should be determined. The study of these two problems takes place in finding evidence for the non-existence of graphs whose number of vertices is very close to the upper limit for the number $n_{\lambda,L}$, called Moore's limit, numerous studies give results about the existence of graphs that improve the lower limit of this number. The main goal of this paper is to study the problem of the existence of Moore graphs, that is, those graphs which, under the given conditions, have the number of vertices equal to Moore's limit. In the introductory section of the paper, the fundamental concepts and basic properties of the Moore graph are precisely defined. Furthermore, all Moore graphs of diameter 2 are described, and special emphasis is placed on the very important Moore graph known as the Petersen graph. Numerous figures are also offered for a better insight into its structure. The last part of the paper is covered by the problem characterizations of Moore graphs of diameter 3.

Keywords: Moore graph, Vertex degree, Diameter of graph, Petersen graph.

Definition of Moore graphs

The term of Moore graphs will be introduced using a special tree called a BFS (Breadth First Search) tree[1]. Therefore, suppose that there is a connected graph with n vertices whose largest degree is λ and whose diameter is L. It is necessary to find the largest number of vertices that can have such a graph. The case $\lambda = L = 1$ is trivial, i.e. the only graph of diameter 1 and highest degree 1 is the complete graph K_2, so $n = 2$. Therefore, let's assume that $\lambda \geq 2$ and $L \geq 2$. Let's construct a BFS tree with vertex-root v.

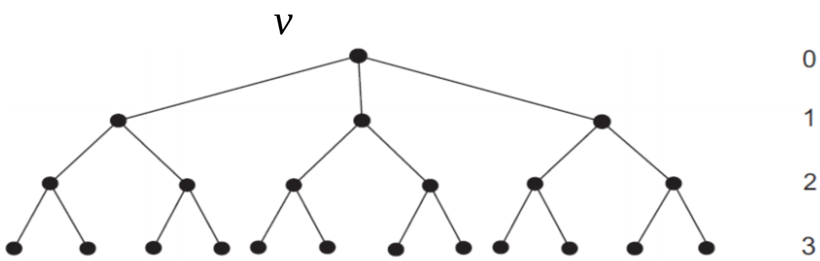

Figure 1: BFS tree when $\lambda = 3$ and $L = 3$

From the vertex v along a path of length 1, we can visit at most λ vertices. From a vertex v along a path of length 2 we can visit at most $\lambda \cdot (\lambda - 1)$ vertices, etc. Finally, from a vertex v along a path of length L we can visit at most $\lambda \cdot (\lambda - 1)^{L-1}$ vertices. See Fig.1. We get:

$$n \le 1 + \sum_{k=1}^{L} \lambda \cdot (\lambda - 1)^{k-1} = 1 + \lambda + \lambda \cdot (\lambda - 1) + \ldots + \lambda \cdot (\lambda - 1)^{L-1}$$

$$= 1 + \lambda \left(1 + (\lambda - 1)^1 + (\lambda - 1)^2 + \ldots + (\lambda - 1)^{L-1}\right) \quad (1)$$

$$= \begin{cases} 1 + \lambda \cdot \dfrac{(\lambda - 1)^L - 1}{\lambda - 2} & , \lambda > 2 \\ 2L + 1 & , \lambda = 2. \end{cases}$$

The number on the right side of inequality (1) is called Moore's limit and denoted by $M_{\lambda,L}$. Edward F. Moore (see [7]) posed the problem of classification and characterization of graphs which, with the given largest degree and diameter, have the number of vertices equal to $M_{\lambda,L}$. Such graphs have special properties and are called Moore graphs. They are necessarily λ-regular. Moore graphs of degree λ and diameter L are said to be Moore graphs of type (λ, L). Initially, the Moore graphs of diameters 2 and 3 were studied (see [10], [12]). In the case of diameter 2, it was shown that the Moore graph exists only for $\lambda = 2, 3, 7$ and possibly 57. For cases of $\lambda = 2, 3$ and 7 proved the uniqueness of the corresponding Moore graphs (see [6], [7]). For $L = 3$ it is proved that there is a unique Moore graph and this is a 7-cycle or C_7 (in this case $\lambda = 2$). These assertions are proved using the eigenvalues and eigenvectors of the adjacency matrix of the graph (and its main submatrices), see [15], [16].

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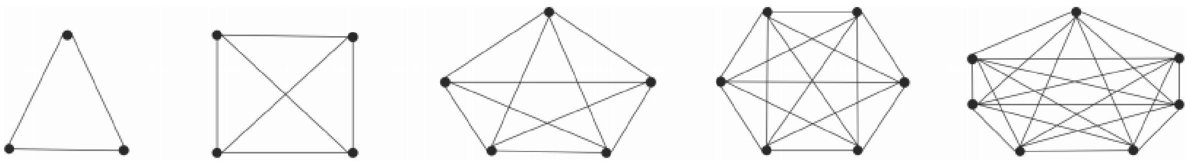
[1] If the largest degree of the graph is given, then by constructing a BFS-tree we get the largest number of vertices of the graph, because each level is filled to the maximum, and only then do we move to the next level.

Basic properties of Moore graphs

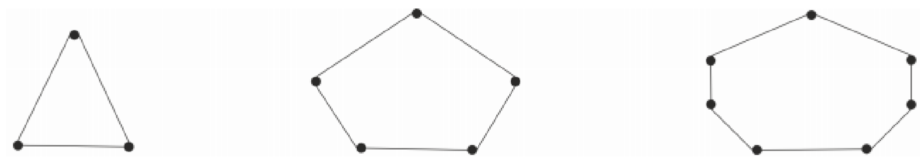
Notice that the trivial graph is a Moore graph of type $(\lambda, L) = (0,0)$. Unique Moore graphs of type $(\lambda,1)$ are complete graphs $K_{\lambda+1}$.



K_3 K_4 K_5 K_6 K_7

Figure 2: Some complete graphs $K_i, (i = 3,4,5,6,7)$

For each $L \in \mathrm{N}$, the cycle C_{2L+1} is a unique Moore graph of type $(2, L)$.



C_3 C_5 C_7

Figure 3: Some cycles $C_i, (i = 3,5,7)$

Let's take an arbitrary Moore type graph (λ, L) and select one of its vertices, which we will consider as a vertex-root. For $k = 0,1,...,L$, we define level k as the set of all vertices at a distance of k from the root, and we define n_k as the number of vertices at level k. Thus, at the zero level there is a unique vertex-root, at the first level there are all vertices at a distance of one from the root, on the second level there are all vertices at a distance of two from the root, etc.

For the number of vertices n_k at level k, it is clear that: $n_0 = 1$ and $n_k \le \lambda \cdot (\lambda - 1)^{k-1}$ for all $1 \le k \le L$. Since the Moore graph is λ-regular with the number of vertices $M_{\lambda,L}$, edges can exist only between level k and $k+1$, $(k = 0,1,...,L-1)$ or between vertices located on level L.

Furthermore, there follows a statement and proof of an important polynomial recursion related to Moore graphs.

Proposition 1. [4] For $\lambda, L \in \mathrm{N} \cup \{0\}$, we define the polynomials $P_j(x)$ recursively as follows:

$$\begin{aligned} P_{j+1}(x) &= xP_j(x) - (\lambda - 1)P_{j-1}(x) \\ P_1(x) &= x + 1 \\ P_0(x) &= 1 \end{aligned} \qquad (2)$$

If A is the adjacency matrix of a Moore graph of type (λ, L), then $P_L(A) = S$, where S is a square matrix with all elements equal to 1.

Proof. For fixed λ and L, assume that there is a Moore graph of type (λ, L) and denote it by $G = (V, E)$. For $0 \le r \le L$ and $u, v \in V$, the following holds:

$$[P_r(A)]_{u,v} = \begin{cases} 1, & \text{if there is a path from } u \text{ to } v \text{ with a maximum length of } r \\ 0, & \text{otherwise}. \end{cases}$$

That means the element in the matrix $P_r(A)$ located in the row corresponding to the vertex u and in the column corresponding to the vertex v is equal to 1 if exists is an (u,v)-path and if $d(u,v) = r$ holds. Otherwise, the corresponding element of the matrix is equal to zero.
The proof of this statement will be realized by the method of mathematical induction by number r. For $r = 0$ we have $P_0(A) = I$ because a path of length 0 exists only between vertices that are identical, so the polynomial P_0 equalized in matrix A gives the unit matrix I.
For $r = 1$, we observe all paths between vertices in the graph G that are of length 0 or 1. It is clear that in that case $P_1(A) = A + I$.
Assume that $2 \le r \le L$. According to the definition of the product of two matrices, we have:

$$[AP_r(A)]_{u,v} = \sum_{w \in V} A_{u,w} [P_r(A)]_{w,v} \qquad (3)$$

Without loss of generality, we can assume that the vertex v is a root. Let the vertex u be at the level i. Let's consider the following cases:

- Case a): When $0 \le i < r$

Given that u is at the level of i, it is known that there is a path from the root v to vertex u, and since $i < r$, according to the assumption of induction, $[P_{r-1}(A)]_{u,v} = 1$ is valid.

Furthermore, is valid $A_{u,w}[P_r(A)]_{w,v} = 1$ if and only if $wu \in E$. If u and w are adjacent, then w is either at the level $i-1$ (this is only valid if $i \ge 1$) or at the level $i+1$. As $d(u,v) = i < r$, it holds that $[P_r(A)]_{w,v} = 1$ so $A_{u,w}[P_r(A)]_{w,v} = 1$.

Conversely, if $A_{u,w}[P_r(A)]_{w,v} = 1$, then both factors are different from zero, i.e. equal to 1, so u and w are adjacent vertices.

There are exactly λ such $w \in V$, so the sum in (3) is equal to λ. Using the definition of the polynomial $P_j(x)$, we conclude that:

$$[P_{r+1}(A)]_{u,v} = \lambda - (\lambda - 1) = 1.$$

□

- Case b): When $r \le i \le r+1$

Now the distance between vertices u and v is at least r, so $[P_{r-1}(A)]_{u,v} = 0$ must be valid. Similar to the first case, is valid $A_{u,w}[P_r(A)]_{w,v} = 1$ if and only if $wu \in E$ and additionally it holds that w is at the level of $i-1$. If w were at the $i+1$ level, then the distance between w and v would be greater than r, so we would have $[P_r(A)]_{w,v} = 0$. However, we have only one such $w \in V$, so the sum in (3) is equal to 1. Therefore,

$$[P_{r+1}(A)]_{u,v} = 1.$$

□

- Case c): When $i > r+1$

The vertex u is more than $r+1$ away from the root v. Hence $[P_{r-1}(A)]_{u,v} = 0$ must hold. But then it is also $A_{u,w}[P_r(A)]_{w,v} = 0$ for every $w \in V$, because if u and w are adjacent, then w is at a level that is at least $r+1$. We get that it is

$$[P_{r+1}(A)]_{u,v} = 0.$$

The proof by induction is complete. Since graph G is of diameter L, between every two vertices there is a path of length less than or equal to L, so $P_L(A) = S$ holds.

□

Moore graphs of diameter 2

Let G be a Moore graph of type $(\lambda, 2)$ and let A be the adjacency matrix of graph G. Then, from formula (1) it follows

$$n = M_{\lambda,2} = 1 + \sum_{k=1}^{2} \lambda(\lambda-1)^{k-1} = 1 + \lambda + \lambda(\lambda-1) = 1 + \lambda^2.$$

For the adjacency matrix A of the graph G, we will describe its spectrum. Let us denote by p a vector whose components are all equal to 1.

Lemma 1. The adjacency matrix of a Moore graph G has exactly 3 distinct eigenvalues. One is λ, and the other two are β_1 and β_2 given by the equations:

$$\beta_1 = \frac{-1+\sqrt{4\lambda-3}}{2}, \qquad \beta_2 = \frac{-1-\sqrt{4\lambda-3}}{2} \tag{4}$$

The eigenvalue λ has a multiple of 1.

Proof. [13] By Proposition 1, the adjacency matrix A satisfies the equation:

$$A^2 + A - (\lambda-1)I = S \tag{5}$$

We notice that $Sp = np$ and $Ap = \lambda p$, so p is an eigenvector of matrix S, but also of matrix A, and λ is an eigenvalue of A. Since the G is a λ-regular graph, the eigenvalue λ of A has multiplicity 1.
Assume that q ($q \neq p$) is an eigenvector of A with a corresponding eigenvalue of β. Due to the orthogonality of the set of all eigenvectors of the symmetric matrix A, it follows that $p^T q = 0$, so we conclude that the sum of all components of the vector q is equal to zero. It follows that $Sq = 0$. Multiplying (5) by $q \neq 0$ we get:

$$A^2 q + Aq - (\lambda-1)q = Sq$$
$$A(Aq) + Aq - (\lambda-1)q = 0$$
$$\beta Aq + \beta q - (\lambda-1)q = 0$$
$$\beta^2 q + \beta q - (\lambda-1)q = 0$$
$$q\left[\beta^2 + \beta - (\lambda-1)\right] = 0$$
$$\Rightarrow \qquad \beta^2 + \beta + 1 - \lambda = 0$$

The obtained quadratic equation has two different real solutions β_1 and β_2 given by (4).

□

Further, we will consider the case when the numbers β_1 and β_2 are irrational, and then the case when they are rational.

Lemma 2. If β_1 and β_2 are irrational numbers, then $\lambda = 2$.

Proof. According to Lemma 1, the sum of multiples of the eigenvalues β_1 and β_2 of the adjacency matrix A is $n-1$. Let a be a multiple of the eigenvalue β_1, and b be a multiple of the eigenvalue β_2. It follows that $a+b=n-1=\lambda^2$. We know that the trace $tr(A)$ of the matrix A is equal to the sum of its diagonal elements, so in our case $tr(A)=0$. However, the trace of the matrix A can also be obtained as the sum of its eigenvalues, so using Lemma 1 we get:

$$a\beta_1 + b\beta_2 + \lambda = 0$$

$$\frac{a}{2}\left(-1+\sqrt{4\lambda-3}\right)+\frac{b}{2}\left(-1-\sqrt{4\lambda-3}\right)+\lambda=0$$

$$\vdots$$

$$a-b=\frac{\lambda(\lambda-2)}{\sqrt{4\lambda-3}}$$

The left side of the last equality is an integer since it is obtained as the difference of two natural numbers. The right side of the same equality is the quotient of an integer and an irrational number (as by assumption β_1 and β_2 are irrational numbers, therefore $4\lambda-3$ is not the square of any natural number). Thus, the last equality makes sense if and only if $a=b$ holds, that is

$\lambda(\lambda-2)=0$.

The case $\lambda=0$ is not possible because the diameter of graph G is equal to 2. We conclude that $\lambda=2$ is valid.

□

The following theorem states and proves the main result of Moore graphs:

Theorem 1. A Moore graph of type $(\lambda,2)$ exists only for $\lambda=2,3,7$ or 57.

Proof. In Lemma 2, the existence of the Moore graph for $\lambda=2$ is confirmed. For other cases, it should be assumed that β_1 and β_2 are rational numbers, that is, that $4\lambda-3=t^2$ is valid, where t is some natural number. Let us denote by k the multiplicity of the eigenvalue β_1. Then according to Lemma 1, the multiplicity of the eigenvalue β_2 is $n-1-k$.

Since $tr(A)=0$, the following holds:

$$\lambda+k\left(\frac{t-1}{2}\right)+(n-1-k)\left(\frac{-t-1}{2}\right)=0.$$

Furthermore, it is valid that $n=\lambda^2$ and $\lambda=(t^2+3)/4$, and after inclusion in the above equation we get:

$$t^5+t^4+6t^3-2t^2+(9-32k)t-15=0. \qquad (6)$$

From equation (6), we take only non-negative integer solutions. Since it is a polynomial of the fifth degree (variable t) with integer coefficients, the integer zero point t must be divided by the free term 15. We get:

$$\begin{array}{llll} t=1, & k=0, & \lambda=1, & n=2 \\ t=3, & k=5, & \lambda=3, & n=10 \\ t=5, & k=28, & \lambda=7, & n=50 \\ t=15, & k=1729, & \lambda=57, & n=3250 \end{array} \tag{7}$$

Since the diameter of G is equal to 2, the case when $\lambda=1$ is not possible, so only cases remain: $\lambda=3,7$ and 57.

□

Moore graph of type (2,2)

The simplest Moore graph of diameter 2 is the cycle C_5. The graph is specific because it is the only cycle that is self-complementary, i.e. $C_5 \cong \overline{C_5}$ is valid.

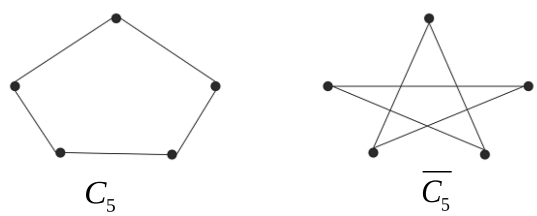


Fig. 4: Cycle C_5 and its complement $\overline{C_5}$

Although they belong to the class of simple graphs, cycles are very interesting graphs because of their adjacency matrix. In the following, we will state and prove a theorem that fully characterizes the spectrum of the adjacency matrix of the cycle C_n.

Theorem 2. The spectrum of the adjacency matrix A of the cycle C_n is set

$$\mu(A)=\left\{2\cos(\frac{2\pi k}{n}) \middle| \ , k=1,2,...,n-1\right\}.$$

Proof. Let Q be a square matrix of order n whose first row is equal to $(0,1,0,...,0)$. Each subsequent row is obtained from the previous one by moving each element one place to the right, and the last element comes first. So, the second row is $(0,0,1,0,...,0)$,…, the penultimate row is $(0,0,...,1)$, and the last row is $(1,0,...,0)$. For an arbitrary integer l, the matrix Q^l is a permutation matrix that has the number one in the first row at position $l+1$, all other elements are zero, and each subsequent row is created from the previous one by moving each element to the right by one position. This makes sense for arbitrary l since for $l>n$ we take numbers modulo l instead of l. The cycle adjacency matrix holds:

$A(C_n) = Q + Q^{-1}$.

Using the eigenvalues of the matrix Q, we will determine the eigenvalues of the matrix $A(C_n)$. If $w = (w_1, ..., w_n)^T$ is an eigenvector of Q corresponding to the eigenvalue δ, then it holds:

$$w_1 = \delta w_n = \delta^2 w_{n-1} = ... = \delta^n w_1.$$

The equation $\delta^n = 1$ has n mutually distinct complex roots which can be written as ω^i where is it $\omega^i = e^{\frac{2\pi i}{n}}$, $(i = 0,1,...,n-1)$.

It follows that the spectrum of the matrix Q^l for arbitrary l is equal to:

$\mu(Q^l) = \{1, \omega^l, \omega^{2l}, ..., \omega^{(n-1)l}\}$.

For the eigenvalue of the matrix C_n, we get:

$A(C_n)u_k = Qu_k + Q^{-1}u_k = (\omega^k + \omega^{-k})u_k$.

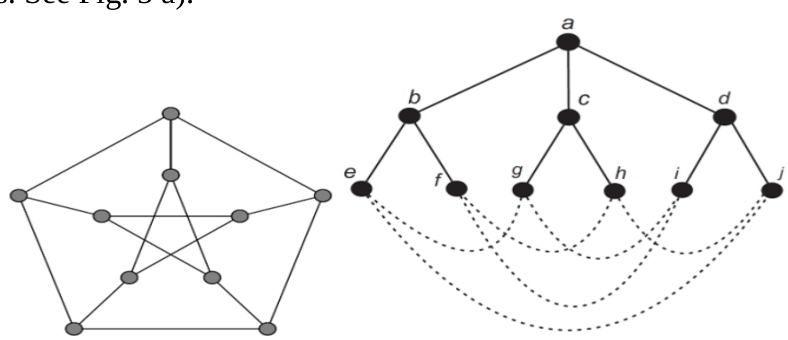
It follows:

$\omega^k + \omega^{-k} = 2\cos(\frac{2\pi k}{n}),\ k = 1,2,...,n-1$.

□

Moore graph of type (3,2)

A Moore graph of type (3,2) is more commonly known as a Petersen graph. The graph is extremely interesting because it very often appears as a counterexample to many statements in graph theory. Below, we will present the Petersen graph using some of its prominent properties. A Petersen graph is a simple 3-regular graph with 10 vertices and 15 edges. See Fig. 5 a).



a) b)

Figure 5: a) The general form of the Peterson graph; b) Construction of the Peterson graph

Proposition 2. Petersen graph is unique.

Proof. [2] First, we construct a BFS tree with 10 vertices as in Figure 5. The edges of the tree are drawn with a solid straight line. All vertices except leaves have degree 3. It is necessary to add 6 more edges between leaves e, f, g, h, i and j, making sure that the resulting graph is 3-regular and that its diameter is equal to 2. This can be done in a unique way. In Figure 5, the new 6 edges are drawn with a broken line. The resulting graph is a Petersen graph.

Through two propositions, we will present two more interesting properties of the Petersen graph, for the proof of which one figure will be sufficient [1].

Proposition 3. A Petersen graph is a complete bipartite graph with 10 vertices.

Proof. Figure 6.

Proposition 4. The Petersen graph is the complement of the line graph of the complete graph K_5.

Proof. Figure 7.

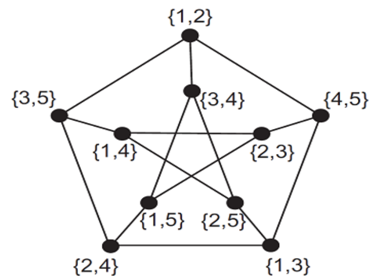


Figure 6: Petersen graph as graph $K(5,2)$

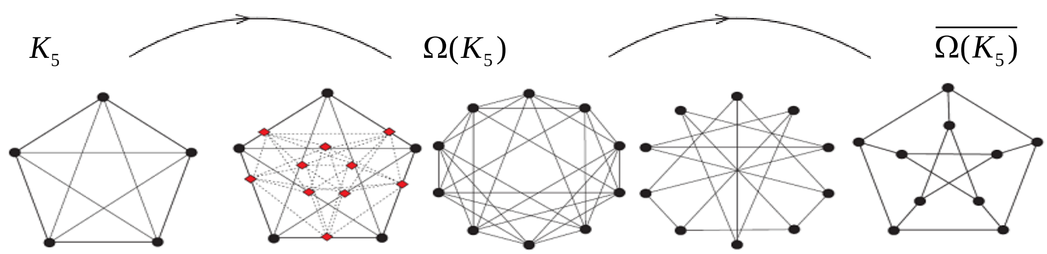


Figure 7: Petersen graph as graph $\overline{\Omega(K_5)}$

Moore graphs of diameter 3

In this paper, the last thing we will prove is that: a Moore graph of type $(\lambda, 3)$ exists if and only if $\lambda = 2$, that is, that the only Moore graph of diameter 3 is cycle C_7.

To prove this claim, we will need the following auxiliary result:

Lemma 3. Let the polynomials $P_j(x)$ be defined as in Proposition 1 and let $\lambda \geq 2$ and $L \geq 2$ hold. If $P_L(x)$ is irreducible over Q, then a Moore graph of type (λ, L) exists only in the case when $\lambda = 2$.

Proof. As in the proof of Lemma 1, if β ($\beta \neq \lambda$) is an eigenvalue of the matrix A then it is immediately shown that β is a root of $P_L(x)$. Considering that $P_L(x)$ is irreducible over Q, that $P_L(x)$ must divide the characteristic polynomial of the matrix A. We conclude that the root of $P_L(x)$ is an eigenvalue of A.

From the construction of the polynomial $P_L(x)$, it is clear that it is a polynomial of degree L and that its coefficients in addition to x^L and x^{L-1} are equal to 1. Let them be $\beta_1, \beta_2, ..., \beta_L$ roots of the polynomial $P_L(x)$ that are different from λ. According to Viet's formulas we get:

$$\sum_{m=1}^{L} \beta_m = -1.$$

Due to the irreducibility of the polynomial $P_L(x)$, the eigenvalue $\beta_1, \beta_2, ..., \beta_L$ must have equal multiples, which we will denote by k. From equation (1), for $\lambda > 2$ we have:

$$n = 1 + \lambda\left(\frac{(\lambda-1)^L - 1}{\lambda - 2}\right).$$

Furthermore,

$$k = \frac{n-1}{L} = \frac{\lambda}{L}\left(\frac{(\lambda-1)^L - 1}{\lambda - 2}\right).$$

Since $tr(A) = 0$, we obtain:

$$0 = \lambda + k\sum_{m=1}^{L} \beta_m = \lambda - k.$$

Using the formulation for number k, we conclude that the following holds

$$(\lambda-1)^L - L(\lambda-1) + (L-1) = 0 \qquad (8)$$

If we consider the left side in (8) as a polynomial in the variable $\lambda - 1$, then due to the condition $L \geq 2$, equation (8) has at most two positive solutions. However, $\lambda = 2$ is a double solution of that equation, so no $\lambda > 2$ can be a solution.

□

Theorem 3. A Moore graph of type $(\lambda, 3)$ exists if and only if $\lambda = 2$.

Proof. By Proposition 1, we have:

$P_3(x) = x^3 + x^2 - 2(\lambda-1)x - (\lambda-1)$.

Assume the opposite, i.e. let there be a Moore graph of type $(\lambda,3)$ for some $\lambda > 2$. By Lemma 3, $P_3(x)$ has a rational root s. Since $P_3(x) \in Z[x]$ and its leading coefficient is 1, we can assume that s is an integer. From $P_3(x) = 0$, we obtain:

$$\lambda - 1 = \frac{s^2(s+1)}{2s+1}.$$

Note that the numbers $2s+1$ and s are mutually prime, and so are the numbers $2s+1$ and $s+1$.
Therefore, we have $2s+1 = \pm 1$ from which we get $\lambda = 1$, which contradicts the assumption $\lambda > 2$.

□

Conclusions

In graph theory, Moore's graphs belong to a special class of graphs due to several applicable properties, and numerous researches are focused on finding Moore's graphs of a certain type[1],[3]. It is worth mentioning that in graph theory there is still an open problem of the existence of a Moore graph of type (57,2), i.e. although it can be calculated numerically, no example of such a graph has yet been found [4], [8], [9]. For that graph, the number of vertices was 3250 and the number of edges was 92625. Furthermore, regarding the problem of the existence of a Hamiltonian path and a Hamiltonian cycle of which connected graphs, a special type of Moore's graph, i.e. Peterson's graph, gives a partial answer to an as yet unproven statement: "Every connected graph that is transitive by vertices contains a Hamiltonian path". In addition to the problem of the existence of Moore's graphs, there is also the problem of the construction of these graphs. The Moore graph of type (7,2) belonging to the class of so-called "strongly regular graphs" was constructed by A. Hoffman and R. Singleton [13].

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