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Professional paper

THE USE OF QUATERNIONS IN THE CALCULATION OF THE SUN'S APPARENT MOVEMENT ACCORDING TO MERCURY AND ITS COMPARISON WITH THE SUN'S APPARENT MOVEMENT ACCORDING TO EARTH

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Abstract

In this paper, the apparent movement of the Sun according to Mercury has been studied and a comparison of this movement has been made with the apparent movement of the Sun according to Earth. The curve of the apparent movement of Mercury is obtained by using quaternions. To achieve this, the celestial sphere is accepted to have a radius of r = 1. The equatorial plane of Mercury is intercepted by its elliptical plane on axis X of the coordinate system. This system coincides with the equatorial coordinate system of Mercury. The apparent movement of the Sun according to Mercury is accepted to begin at point (1, 0, 0). The curve drawn by this point is calculated by using quaternions as rotation operators. For both the daily and yearly apparent movements of the Sun according to Mercury, a quaternion each is defined. These quaternions are used to produce rotation operators for each movement. Afterward, a comparison is made between this curve and the curve produced by the apparent movement of the Sun according to Earth. This paper, in which the discipline of mathematics joins that of astronomy, helps present the usefulness of quaternions as rotation operators and simultaneously helps new astronomers perceive the apparent movement of the Sun on other planets.

Keywords: Spherical Spiral, Quaternions, Apparent Movement of the Sun, Rotational Motion.

1. Introduction

The possibility of the existence of life on other planets has always intrigued and continues to intrigue mankind. There has been no evidence of life in the universe except on planet Earth. However, the curiosity about what life could be like on other planets considering their physical conditions can be satisfied by gathering data on these planets.

One of these data points is the apparent movement of the Sun. As it is known, this movement consists of two different apparent movements of the Sun. The first one is the daily and the second is the yearly apparent movement of the Sun. The first movement enables the creation of the night and day. The second movement, combined with the inclination of the earth's axis enabling the formation of an angle between the equatorial and elliptical celestial plane, enables the formation of the seasons. Both of these movements are also conditioned by the speed of the Earth's motion around itself and the Sun. These movements exist in the other planets of the solar system as well but the parameters that enable these movements are different. This is reflected in visible changes in the length of day and night, and the formation of seasons on the other planets.

Another aspect that has become part of this work is the calculation of these movements by using quaternions. As it is known, quaternions are very useful when it comes to the calculation of the rotational motions in 3-dimensional spaces.

This paper seeks to answer the question of what the apparent movement of the Sun would be like during a Mercurian year in the sky of Mercury. To calculate this curve the rotation operators produced by quaternions have been used. This calculation method has been previously used to calculate the apparent movement of the Sun according to planet Earth (Güçler et al. 2022). By finding the curve of the apparent movement of the Sun according to Mercury, the opportunity is created for a comparison to be made between it, and the curve of the apparent movement of the Sun according to Earth.

To understand and present the problem the author has benefited from the references (Güçler et al. 2022; Karaali 1985; Kızılırmak 1977; Kummer 1996; Lowenstein 2012; Motz and Duveen 1966; Woolard and Clemence1966). The details about the quaternions can be viewed from the references (Altman 1986; Delphenich 2012; Hacısalihoğlu 1983; Kuipers 1975; Kuipers 1998; Griffin 2017; Dong et al. 2020). The information needed for the other calculations is found in the references (Fisher and Ziebur 1965). The main idea that gave rise to this paper was first presented at the 5th International Conference of Natural Sciences and Mathematics - University of Tetova by the authors of this paper (Güçler 2023).

2. Preliminaries

2.1. Quaternion's algebra: A quaternion is a hyper-complex number. The most important rule of this invention of Hamilton is:

$$i^2 = j^2 = k^2 = ijk = -1 \; and \; ij = k = -ji, jk = i = -kj, ki = j = -ij \qquad (1)$$

i, j, and k are the components of the vector part of the quaternion and they will be used to represent the standard orthogonal base of R^3.

The quaternion can be thought of as a quadruple of real numbers as it is shown below.

$$q = q_0 + \alpha = q_0 + iq_1 + jq_2 + kq_3, \; \alpha = iq_1 + jq_2 + kq_3$$
(2)

where m_0 is the scalar part and α is the vector part.

Multiplication of quaternions is done according to the following rule

For $q = q_0 + \alpha_q = q_0 + iq_1 + jq_2 + kq_3$ and $p = p_0 + \alpha_p = p_0 + ip_1 + jp_2 + kp_3$

$$q\,p = \left(q_0 + \alpha_q\right)\left(p_0 + \alpha_p\right)$$
(3)
$$= q_0 p_0 - \langle \alpha_q, \alpha_p \rangle + q_0 \alpha_p + n_0 \alpha_q + \alpha_q \, \Lambda \, \alpha_p$$

"$\langle , \rangle$" represents the scalar product of vectors, and "Λ" represents the cross product of vectors.

The complex conjugant of $q = q_0 + iq_1 + jq_2 + kq_3$ is $q^* = q_0 - iq_1 - jq_2 - kq_3$
(4)

Definition: The quaternion whose scalar part is zero is called a pure quaternion.

The quaternion that will be used as a rotation operator is:

$q = q_0 + \alpha = \cos\varphi + u \sin\varphi$ and $q^* = q_0 - \alpha = \cos\varphi - u \sin\varphi$
(5)

where $u = \alpha / \ |\alpha| = \alpha / sin\,\varphi$

Theorem 1: For any $q \ = \ q_0 + \boldsymbol{q} = \ cos\,\varphi \ + \ u\,sin\,\varphi$ unit quaternion (where q_0 is the scalar part and **q** is the vector part of the quaternion) and for any vector $v\,\epsilon\,R^3$ the action of the operator

$$L_q(v) \ = \ q \ \times \ v \ \times \ q^*$$

on v may be interpreted geometrically as a rotation of the vector v through an angle 2φ about **q** as the axis of the rotation. (Kuipers 1998)

Theorem 2: Suppose that q and p are unit quaternions that define the quaternion rotation operators:

$$L_q(u) \ = \ q \ \times \ u \ \times \ q^* and \ L_p(v) \ = \ p \ \times \ v \ \times \ p^*$$

Then the quaternion product pq defines a quaternion operator L_{pq} which represents a sequence of operators, L_q followed by L_p. The axis and the angles of rotation are those represented by the quaternion product, $r \ = \ p \ \times \ q$. (Kuipers 1998)

2.2 Apparent Movement of the Sun According to Earth: As it is known, the Earth rotates every day in a positive motion around its own axis and parallel to the equatorial plane. This motion is reflected as the apparent movement of the Sun, occurring in the negative direction.

It is also known that the Earth rotates every year around the Sun in the positive direction, in an ecliptic orbit found in the ecliptic plane. However, this motion makes it appear as if it is the Sun moving around the Earth, during the year in a positive direction (Figure 1). The ecliptic plane intersects with the celestial equatorial plane and creates a $23^0 27^{'}$ angle. (Karaali 1985)

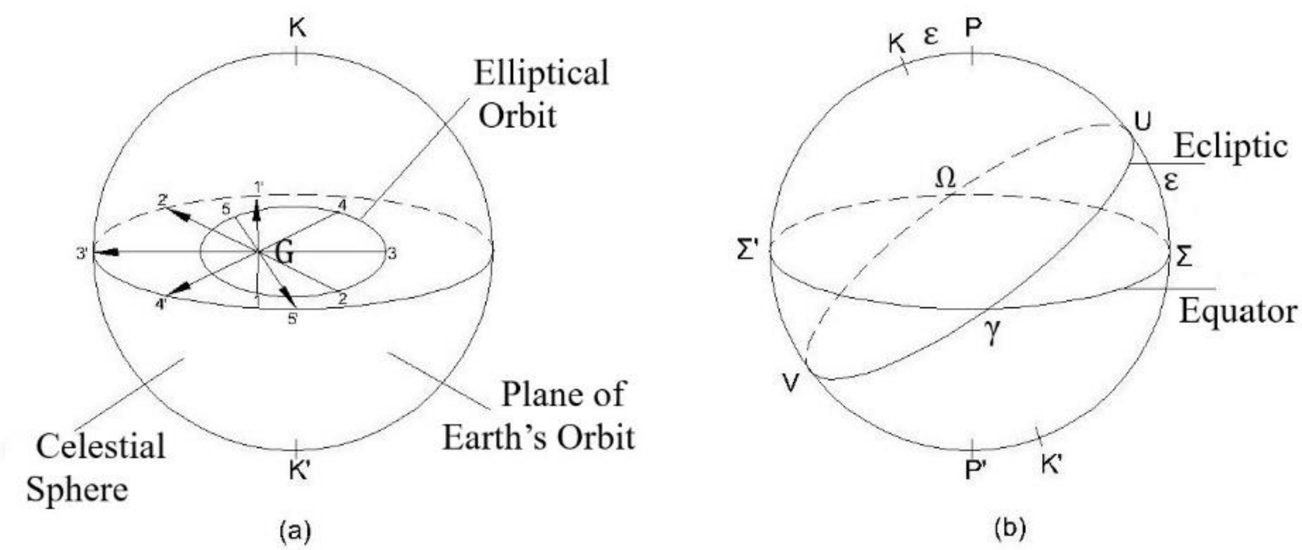

Figure 1. The elliptical orbit made by the actual motion of the Earth (a) the elliptical circle made by the annual motion of the Sun (b)

2.3. The Relationship Between Latitude and the Pole Height of a Point on the Earth: In Figure 2, the latitude e of a C location on the Earth is shown. AB is the Earth's equator and P_1P_1' is its axis. The North P pole of the celestial sphere is on the extension of OP_1. The height of the pole P at a given place is equal to the angular distance of P from the horizon of that place.

This height is indicated by h_p in Figure 2. On the other hand, since the dimensions of the Earth are negligible compared to the dimensions of the celestial sphere, the pole P can be considered on the parallel line drawn from C to OP_1.

So $\widehat{DCQ}$ is equal to pole height h_p. Also, the angles $\widehat{DCQ}$ and $\widehat{COB}$ are congruent. It follows that the angles h_p and e are equal. So, the pole height at a given point on the Earth is equal to the latitude of that place. In Figure 3, examples for two different latitudes are given. (Kızılırmak 1977)

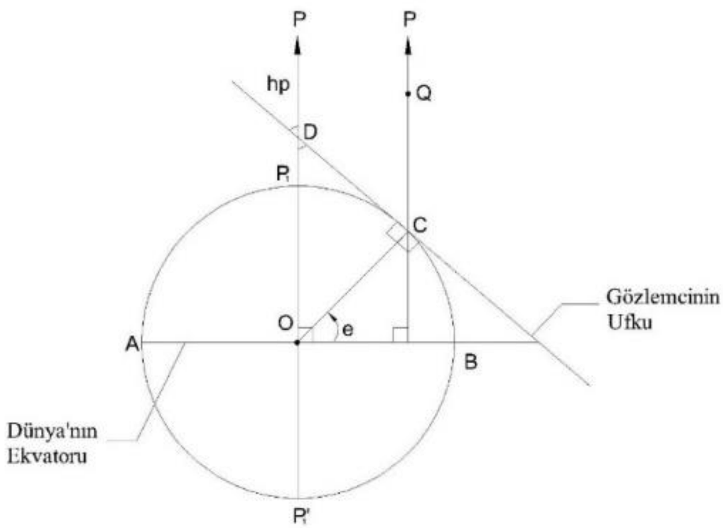

Figure 2. The pole height of any given point on the Earth is equal to the latitude of that point (h_p = e).

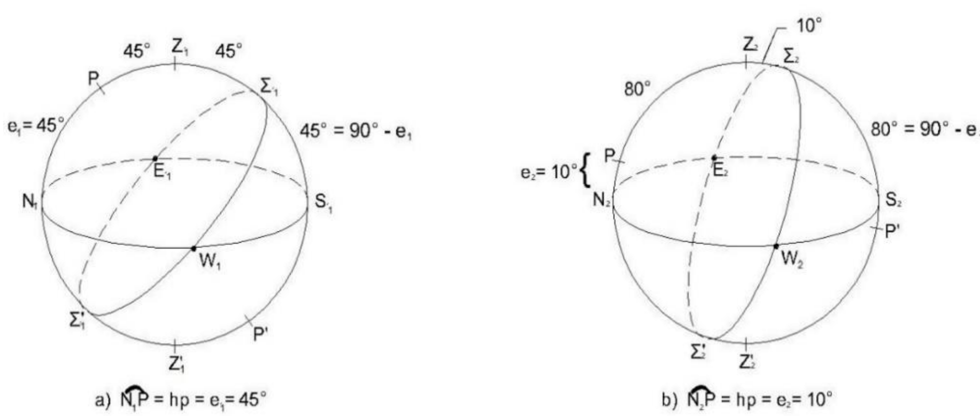

Figure 3. The celestial sphere for 45^0 and 10^0 northern latitudes

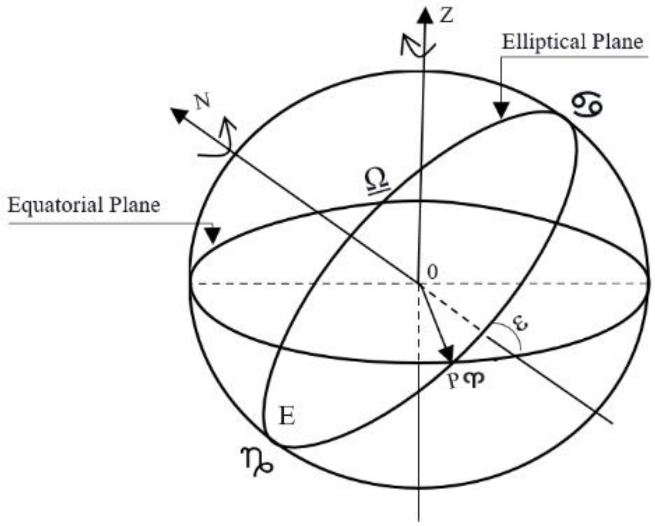

Figure 4. The system in which the apparent movement of the Sun occurs

2.4 The Equation of the Curve of the Apparent Movement of the Sun According to Earth: In this presentation, it is assumed the apparent movement of the Sun occurs in ideal conditions. So, it will be accepted that the apparent movement of the Sun in the ecliptic plane occurs in a circular orbit with a constant angular velocity. Now let plane E represent the ecliptic plane while plane XY represents the plane of the celestial equator, and angle ε represents the angle $\varepsilon = 23^0 27'$ (Figure 4). In this case, the point $(0, 0, 0)$ represents the Earth. In addition, the positive direction of axis X will represent the Aries constellation.

Let Q_1 be the quaternion that will realize the movement in the positive direction around axis N. Let ${Q_2}^*$ be the quaternion that will realize the movement in the negative direction around axis Z. The starting point of the movement is $P = (1, 0, 0)$. The vector OP is $v = (1, 0, 0)$. First, let this vector be transferred to the quaternion space so:

$v = (1, 0, 0)\ vector \rightarrow w = 0 + i + 0j + 0k = i$ corresponds to a pure quaternion.

The first rotation movement will be realized around axis $u = -j\ sin\varepsilon + k\ cos\varepsilon$ with θ angle. The second rotation movement will be realized around axis k with a (cθ) angle in a negative direction. In this case, the Q_1 and ${Q_2}^*$ quaternions that will operate as rotation operators are:

For $a = sin\ \varepsilon$ and $b = cos\ \varepsilon$

$Q_1 = cos\ \theta/2 - j\ asin\ \theta/2 + k\ bsin\ \theta/2$ and ${Q_2}^* = cos(c\theta)/2 - k\ sin(c\theta)/2$

(6)

According to Theorem 2 and if ${Q_2}^* \times Q_1 = Q$ and $w = i$ then

$L_{Q_2^* Q_1}(w_1) = Q \times i \times Q^*$

(7)

So, the calculations are as such:

$$Q = Q_2^* \times Q_1 = (cos(c\theta)/2 - k\ sin(c\theta)/2) \times \left(cos\ \theta/2 - j\ asin\ \theta/2 + k\ bsin\ \theta/2\right) \tag{8}$$

Accordingly, when the rotation operator produced by the Q quaternion is applied to $w = (i,\ 0,\ 0)$, the pure quaternion $W = (\ W_1,\ W_2,\ W_2)$ is obtained as shown below.

$$W = Q \times i \times Q^* \tag{9}$$

When the calculations are made

$$W_1 = (cos(c\theta)cos\ \theta + bsin(c\theta)\ sin\theta)i,\ W_2 = (bcos(c\theta)sin\ \theta - sin(c\theta)cos\theta)\,j,\ W_3 = \left(asin\theta\right)k \tag{10}$$

is found. As a result of transferring pure quaternion W to the vector space, $V = (V_1,\ V_2,\ V_3)$ is obtained.

For $0 \leq \theta \leq 2\pi$, $c = 365,25$ (365.25 the number of days in a year), $a = \sin 23^0 27' = 0,40$ and $b = cos\ 23^0 27' = 0,89$

$$\begin{aligned} V_1 &= X = cos\ \theta\ cos\ (365,25\theta) + 0,89\ sin\ \theta\ sin\ (365,25\theta) \\ V_2 &= Y = 0,89\ sin\ \theta\ cos\ (365,25\theta) - cos\ \theta\ sin\ (365,25\theta) \\ V_3 &= Z = 0,40\ sin\ \theta \end{aligned} \tag{11}$$

This is the equation of the curve of the Sun’s apparent movement according to Earth. (Güçler et al. 2022)

3. The Equation of the Curve of the Apparent Movement of the Sun According to Mercury and its Comparison with the Sun’s Apparent Movement According to Earth

In the following part, the finding of the equation of the apparent movement of the Sun according to Mercury as well as its comparison with the apparent movement of the Sun according to Earth have been taken into consideration.

3.1 The Equation of the Curve of the Apparent Movement of the Sun According to Mercury: Mercury, like all other planets in the solar system, moves around the Sun in an elliptical orbit in the elliptical plane according to Mercury. The angle between the elliptical plane of Mercury and the equatorial plane of Mercury is $\varepsilon = 0^0 2,04'$. In this presentation, this orbit is accepted as circular and the angular velocity in this orbit is accepted as a constant. The planet Mercury completes this movement in the positive direction. From the point of view of an observer on Mercury, this movement appears to be performed by the Sun. In concordance with the apparent movement of the Sun in the elliptical plane around Earth, this apparent movement is completed by the Sun in the positive direction. One full rotation in this orbit is accepted as one Mercurian year.

The second movement of Mercury is the one it performs around its own axis parallel with the Mercurian equatorial plane. This rotation, in concordance with the apparent daily movement of the Sun according to Earth, occurs in the positive direction therefore the apparent movement of the Sun occurs in the negative direction. During a Mercurian year, Mercury completes 0,5 rotations around its axis. So, in a Mercurian year, there are approximately 0,5 Mercurian days.

In concordance with the apparent movement of the Sun according to Earth;

For $0 \leq \theta \leq 2\pi$, $c = 0,5$, $a = sin\ 0^0 2,04' \sim 0$ and $b = cos\ 0^0 2,04' \sim 1$

$$V_1 = X = \cos\theta \cos(0{,}5\theta) + \sin\theta \sin(0{,}5\theta)$$
$$V_2 = Y = \sin\theta \cos(0{,}5\theta) - \cos\theta \sin(0{,}5\theta) \quad (12)$$
$$V_3 = Z = 0$$

This is the equation of the curve of the Sun's apparent movement according to Mercury.

3.2 Comparison Between the Apparent Movement of the Sun According to Mercury with the Sun's Apparent Movement According to Earth: By finding equation (12), the opportunity to compare the graph created by this equation with the one created by equation (11) arises. If the graphic of the equation (11) was drawn, this curve would cover the entirety of the sphere found between the planes $z = -\sin 23^0 27'$ and $z = \sin 23^0 27'$ because the constant c is $c = 365{,}25$. For this reason, to be able to comprehend the shape of the curve, $c = 12$ is chosen instead of $c = 365{,}25$. On the other hand, since the planet Mercury makes one complete rotation around itself in two years, to see this movement in its entirety, the value of angle θ is chosen as $0 \leq \theta \leq 4\pi$. Based on these accepted changes, the graphic shown in Figure 5 is obtained.

The graphs that are obtained, are placed in the sky of each planet in the way they would be seen during a year, on the horizon of an observer that is found at one of the poles of each planet. On planet Earth, the Sun stays at the poles for half a year, and on planet Mercury, the Sun stays for a whole year. The maximal altitudes from the horizon of each planet, however, have a difference of approximately $\sin 23^0 25'$.

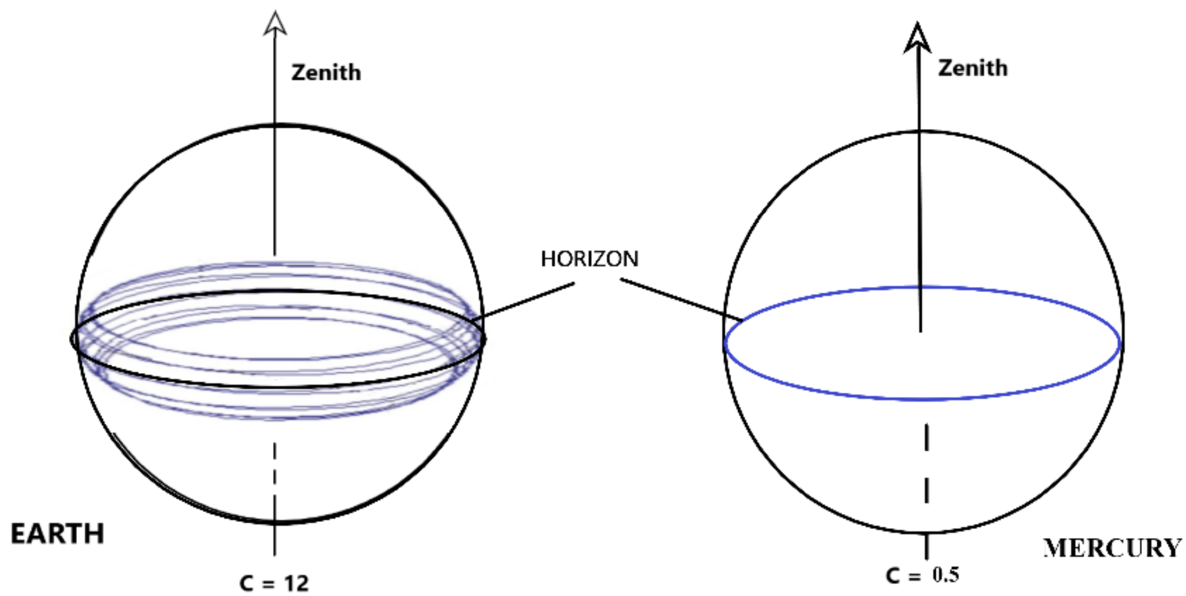

Figure 5. The curve of the apparent movement of the Sun for Latitude = 90^0

In Figure 6, the graphs that are obtained, are placed in the sky of each planet in the way they would be seen during a year on Earth and two years on Mercury, on the horizon of an observer that is found on the equator of each planet. The rays of the sun are perpendicular to the horizontal plane on both planets. The duration of the days and nights is equal. However, because a year on Earth has approximately 365 Earth days and a year on Mercury has only 0.5 Mercurian days, the difference in temperature between days and nights is greater on Mercury.

In Figure 7, the graphs that are obtained, are placed in the sky of each planet in the way they would be seen during a year for Earth and during two years for Mercury, on the horizon of an observer located at a latitude of 45^0 on each planet. The duration of nights and days varies during the year on planet Earth. In contrast, on planet Mercury, these durations hardly change at all. The plane of sun rays that reach an observer during a day, changes over the days of the year, on Earth. However, on Mercury, nothing changes. These two data points show that on Mercury, seasons do not exist as they exist on Earth.

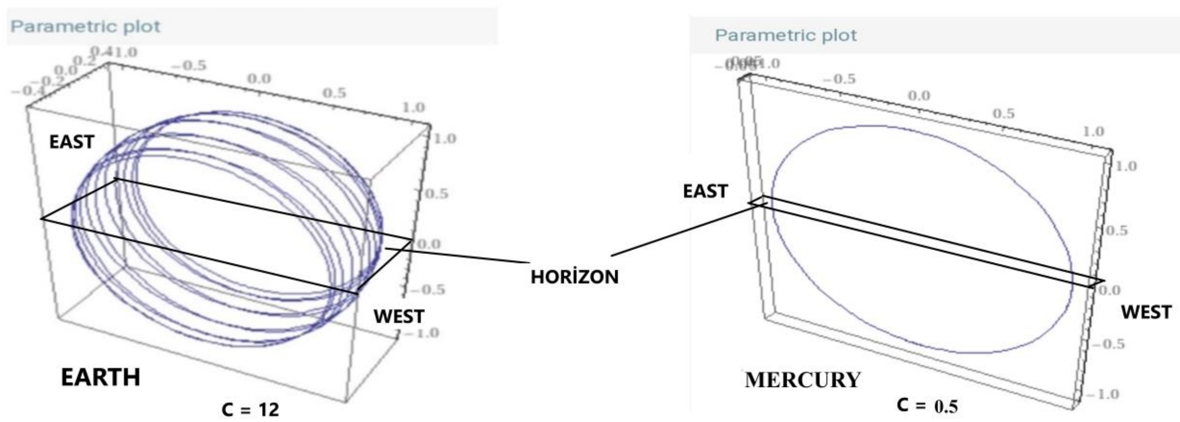

Figure 6. The curve of the apparent movement of the Sun for Latitude = 0^0

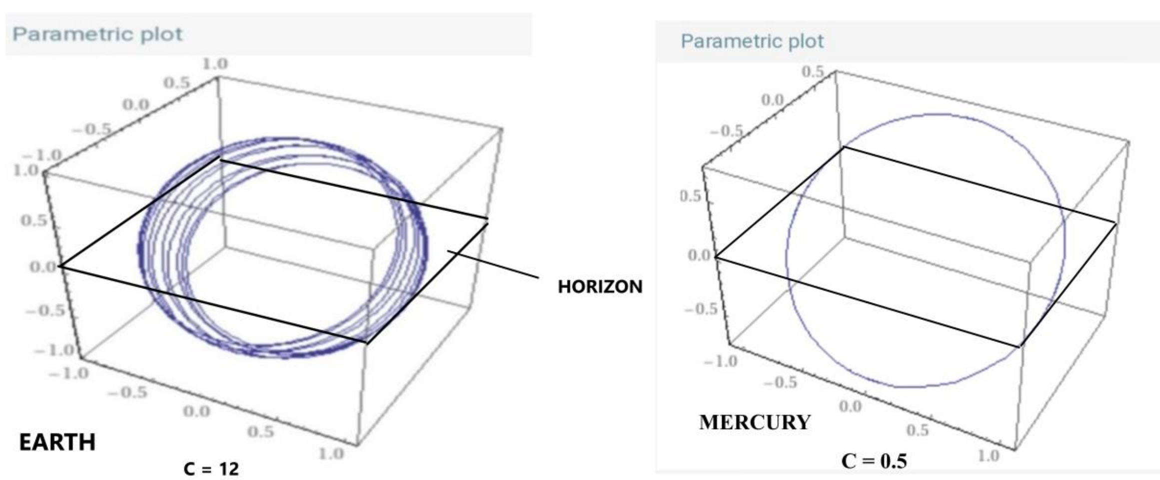

Figure 7. The curve of the apparent movement of the Sun for Latitude = 45^0

4. Conclusion

To conclude, based on the graphs of Figure 5 – Figure 7, it can be said that a day in Mercury is two Mercurian years long. The temperature changes between night and day are immense. The length of night and day is the same on the entire planet. Seasons do not exist in the way they do on Earth.

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