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Professional paper

CHAIN OF A SET IN A COVERING AND CHAIN COMPONENTS UP TO A COVERING

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Abstract

A chain in the open covering $\mathcal{V}$ of a topological space X that joins $U \in \mathcal{V}$ and $V \in \mathcal{V}$ is a finite sequence of elements of $\mathcal{V}$ such that U is the first member, V is the last member and every two consecutive members of the sequence have a nonempty intersection. By $chainV, V \in \mathcal{V}$ it is meant the union of all elements of the covering for which there are chains joining them with V and $chain\mathcal{V}$ is the set that consists of all sets $chainV$ for each $V \in \mathcal{V}$.
A chain in $\mathcal{V}$ that joins $x \in X$ and $y \in X$ is a finite sequence of elements of $\mathcal{V}$ such that x is contained in the first element of the sequence, y is contained in the last element and every two consecutive elements of the sequence have a nonempty intersection. A $\mathcal{V}$-chain component of an element $x \in X$, $Ch(x, \mathcal{V})$, is the set that consists of all $y \in X$ such that there exists a chain in $\mathcal{V}$ that joins x and y.
We prove that $chainV = Ch(x, \mathcal{V})$ for any $V \in \mathcal{V}$ and any $x \in V$, hence $chain\mathcal{V}$ consists of $\mathcal{V}$-chain components. As a consequence, chain connectedness is characterized using the $chainV$ notion.

Keywords: Chain, Star of a set, Open covers, Chain connectedness.

1. Introduction

In [3] by using the notion of chain and coverings of a topological space, it is characterized by the connectedness of the space. In [2] and [4] in a similar way the notion of chain connectedness of a set is introduced. In [1] the notion of a chain of a set in a covering is introduced.

Here we characterize and give some equivalences for some notions used in this literature.

1.1. $\mathcal{V}$*-chain relation:* Let X be a topological space and $\mathcal{V}$ be an open covering of the space X.
For any two points $x, y \in X$ we say that they are $\mathcal{V}$*-chain related* if there exists a finite family $V_1, V_2, \ldots, V_n$ in $\mathcal{V}$ such that $x \in V_1, y \in V_n$ and $V_i \cap V_{i+1} \neq \emptyset, \forall i \in \{1, 2, \ldots, n-1\}$. We denote that with $x \underset{\mathcal{V}}{\sim} y$.

The family $\mathrm{V}_1, \mathrm{V}_2, \ldots, \mathrm{V}_n$ is called *a chain in* $\mathcal{V}$ *joining* x *and* y.

The relation "$\underset{\mathcal{V}}{\sim}$" is an equivalence relation in X. For a fixed $x \in X$, the class represented by x will be denoted by $Ch(x,\mathcal{V})$ and will be called a $\mathcal{V}$-chain component of x. In [2] it is proved that $\mathcal{V}$-chain components are clopen sets.

1.2. Chain of a set in a covering: In [1] it is introduced the notion of chain between two elements of the cover.
A *chain in* $\mathcal{V}$ *from* $U \in \mathcal{V}$ *to* $V \in \mathcal{V}$ is any sequence $V_1, V_2, \dots, V_n$ in $\mathcal{V}$ such that $U \cap V_1 \neq \emptyset, V \cap V_n \neq \emptyset$ and $V_i \cap V_{i+1} \neq \emptyset, \forall i \in \{1,2,\dots,n-1\}$.

For $V \in \mathcal{V}$ by $chainV$ it is denoted the set

$$chainV = \bigcup \{W \in \mathcal{V} \mid \text{there exists a chain in } \mathcal{V} \text{ from } V \text{ to } W\}$$

and by $chain\mathcal{V}$ the set $chain\mathcal{V} = \{chainV \mid V \in \mathcal{V}\}$.

1.3. Star of a set in a covering: For $V \in \mathcal{V}$, by *star of* V we mean the set

$$st(V,\mathcal{V}) = \bigcup \{W \in \mathcal{V} \mid V \cap W \neq \emptyset\},$$

by *star of* V *of degree* n $(n > 1)$ the set defined by the recursive formula $st^n(V,\mathcal{V}) = st(st^{n-1}(V,\mathcal{V}),\mathcal{V})$, and the *infinite star of* V the set $st^{\infty}(V,\mathcal{V}) = \bigcup_{n=1}^{\infty} st^n(V,\mathcal{V})$.

2. $\mathcal{V}$-chain components and $chainV$ sets

In the following theorem we claim that $chainV$ sets are $\mathcal{V}$-chain components.

Theorem 2.1. Let $V \in \mathcal{V}$ and $x \in V$. Then $chainV = Ch(x,\mathcal{V})$.

Proof. Let $y \in chainV$. Then $y \in W$ for some $W \in \mathcal{V}$ such that there exists a chain $V_1, V_2, \dots, V_n$ in $\mathcal{V}$ from V to W. Then $V, V_1, V_2, \dots, V_n, W$ form a chain in $\mathcal{V}$ from x to y, hence $chainV \subseteq Ch(x,\mathcal{V})$.
Now, let $y \in Ch(x,\mathcal{V})$. Then there is a chain in $\mathcal{V}$ from x to y, thus there exists $W \in \mathcal{V}$ a neighborhood of y such that there is a chain from V to W. This implies that $Ch(x,\mathcal{V}) \subseteq chainV$. □

The following corollaries are a direct consequence of Theorem 2.1.

Corollary 2.1. $chain\mathcal{V}$ consists of all $\mathcal{V}$-chain components of X.

Corollary 2.2. Let $V \in \mathcal{V}$ and $x \in chainV$. Then $chainV = Ch(x,\mathcal{V})$.

The set $chainV$ can also be expressed using the infinite star of V in $\mathcal{V}$.

Proposition 2.1. $chainV = st^{\infty}(V,\mathcal{V})$ for all $V \in \mathcal{V}$.

As a consequence, we have:

Corollary 2.3. Let $V \in \mathcal{V}$ and $x \in st^{\infty}(V, \mathcal{V})$. Then $st^{\infty}(V, \mathcal{V}) = Ch(x, \mathcal{V})$.

2.1. $\mathcal{V}$*-chain connectedness and chain*$\mathcal{V}$*-nearness:* For any two points $x, y \in X$ and for any covering $\mathcal{V}$ of X, we say that x and y are $\mathcal{V}$*-near* if there exists $V \in \mathcal{V}$ such that $x, y \in V$.

Proposition 2.2. Two points x and y are $\mathcal{V}$-chain related if and only if they are *chain*$\mathcal{V}$-near.
Proof. Let x and y be $\mathcal{V}$-chain related. Then, from Theorem 2.1, $x, y \in chainV$ for some $V \in \mathcal{V}$, hence they are *chain*$\mathcal{V}$-near.
Conversely, let x and y be *chain*$\mathcal{V}$-near. Then there is $chainV \in chain\mathcal{V}$ such that $x, y \in chainV$, thus, using Theorem 2.1, x and y are $\mathcal{V}$-chain related. □

3. Chain connectedness and *chainV* sets

With the help of $\mathcal{V}$-chain relation, in [2] and [4] it is introduced the notion of chain connectedness. Here we express this notion using *chainV* sets.

3.1. Chain relation and chain$\mathcal{V}$*-nearness:* For any two points $x, y \in X$ we say that they are *chain related in* X if $x \underset{\mathcal{V}}{\sim} y$ for any covering $\mathcal{V}$ of X. We denote that with $x \sim y$.

The relation "$\sim$" is an equivalence relation in X. For a fixed $x \in X$, the class represented by x will be denoted by $Ch(x)$ and will be called *a chain component of* x. Chain components are closed sets. In [3] it is proved that chain components coincide with quasicomponents of the space.

We show that chain relation can be expressed with the help of the notion of *chain*$\mathcal{V}$-nearness.

Proposition 3.1. Two points x and y are chain related if and only if they are *chain*$\mathcal{V}$-near for all coverings $\mathcal{V}$ of X.
Proof. A direct consequence of Proposition 2.2. □

As a consequence, the chain components are expressed by using *chainV* sets.

Corollary 3.1. For any $x \in X$ we have
$Ch(x) = \bigcap_{\mathcal{V}} \{chainV \mid V \in \mathcal{V}, x \in V\}$.

3.2. Chain connected sets and chainV sets: For a given set $C \subseteq X$ we say that it is *chain connected in* X if any two points $x, y \in C$ are chain related in X.

In [2] and [4] it is proved that any connected set is chain connected in the space, but the converse does not hold.

Proposition 3.2. If C is chain connected in X, then $C \subseteq Ch(x)$ for any $x \in C$.

In [2] chain connected sets are characterized using the infinite star of a singleton.

Theorem 3.1. The set C is chain connected in X if and only if for every $x \in C$ and every covering $\mathcal{V}$ of X, $C \subseteq st^{\infty}(x, \mathcal{V})$.

Using this theorem and previous results, we have the following corollaries.

Corollary 3.2. The set C is a chain connected in X if and only if for every $x \in C$ and every covering $\mathcal{V}$ of X, $C \subseteq chainV$, where $V \in \mathcal{V}$ is a neighborhood of x.

Corollary 3.3. The set C is chain connected in X if and only if for every $x, y \in C$ and every covering $\mathcal{V}$ of X, x and y are $chain\mathcal{V}$-near.

4. Conclusions

The notion of chain is a good tool for proving results related to connectedness. Here we showed that some notions defined with the help of chains, used in different articles, have equivalent meanings. We proved that $\mathcal{V}$-chain components coincide with $chainV$ sets and $st^{\infty}(V, \mathcal{V})$, for any member V of the covering $\mathcal{V}$ of the space. Hence, the notion of chain connectedness was characterized using $chainV$ sets and star infinity of members of coverings.

References

[1]. Buklla, Abdulla, and Gjorgji Markoski. 2019. Proximately Chain Refinable Functions. *Hacettepe Journal of Mathematics and Statistics* 48: 1437–42.

[2]. Misajleski, Zoran, N. Shekutkovski, and A. Velkoska. 2019. Chain Connected Sets in a Topological Space. *Kragujevac Journal of Mathematics* 43: 575–86.

[3]. Shekutkovski, Nikita. 2016. On the Concept of Connectedness. *Mat. Bilten* 50: 5–14.

[4]. Shekutkovski, Nikita, Zoran Misajleski, and Emin Durmishi. 2019. Chain Connectedness. *AIP Conference Proceedings*. Vol. 2183: 030015-1-030015-4.