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Professional paper

RK4 AND VECTORIAL RK4 - USED TO ANALYZE A MODIFIED FORM OF FRACTIONAL ORDER LORENZ SYSTEM

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Abstract

The dynamics of a generalized form of fractional-order Lorenz system are investigated by employing a modified version of the Runge-Kutta 4 method (RK4). The method is very simple and very effective for solving differential equations of fractional order, it may be used. To illustrate the new technique, the numerical algorithm is applied in the 3D solution of the Lorenz system by adding the fourth varied parameter, considered as a highly simplified model for the weather.
Parameter fixed dynamical analysis method and chaos diagram are used. Results show that the fractional order Lorenz system has rich dynamical behavior and it is a potential model for application. Investigation of dynamics is realized by fixing the parameters $a=10,\ b=8/3,\ c=24.74$ (system has chaotic behavior, numerically illustrated), for $d\in[-1,\ 10]$, implemented with the aid of Mathematica symbolic package. The fractional derivatives are described in the Caputo sense. Based on RK4 and Vectorial RK4 algorithms, is shown that the system has rich dynamical characteristics, it changes from a non-chaotic system to a chaotic one, which is more complex for smaller fractional derivative order $v\in(0,1)$, closer to 0.

Keywords: Caputo fractional derivative, Lorenz system, Runge-Kutta 4, Vectorial Runge-Kutta 4, dynamical behavior.

1. Introduction

Fractional calculus is not a new topic, it has almost the same history as that of classical calculus. It can be dated back to Leibniz's letter to L'Hˆopital, dated 30 September 1695, in which the meaning of the one-half order derivative was first discussed with some remarks about its possibility (Ross, 1975). So fractional calculus also means fractional integration and fractional differentiation. Different from the typical derivative, there are more than six kinds of definitions of fractional derivatives, not mutually equivalent. The Caputo derivative is defined based on fractional integral and used in this paper (Milici, et al., 2019). Is analyzed numerically the behavior of a dynamical system, which exhibits high sensitivity to initial conditions, known as chaos.
This behavior was first observed by Edward Lorenz while solving a system of three differential equations governing weather prediction on a computer (Lunch, 2007; Li, & Zeng, 2015).

Definition 1.1. (Miller, & Ross, 1993). The Caputo fractional derivative of a function $f(t)$, of order $0<v\le 1$ is defined by:

$$\overline{D}_t^v f(t)=\frac{1}{\Gamma(n-v)}\int_0^t (t-\tau)^{v-1} f^{(n)}(\tau)d\tau \tag{1.1}$$

Is chosen Caputo fractional derivative, because it allows initial conditions to be included in the formulation of the problem.

Definition 1.2. (Milici, et al., 2019; Miller, & Ross, 1993). An *n*-dimensional fractional-order system with Caputo derivatives is described with:

$$\overline{D}^{v}\mathbf{y} = \mathbf{f}\left(\mathbf{y}\right) \tag{1.2}$$

with $v = \left[v_1, v_2, ..., v_n\right]^T$, $0 < v_i \leq 1 \left(i = 1, 2, ..., n\right)$ and $\mathbf{y} \in \mathbb{R}^n$. Equilibrium points $E^*\left(y_1^*, y_2^*, ..., y_n^*\right)$ of (1.2) are the solutions of the equation $\mathbf{f}\left(\mathbf{y}\right) = 0$.

Definition 1.3. (Miller, & Ross, 1993). The trajectory $y\left(t\right) = 0$ of (1.2) is said to be **stable** if for any initial conditions $y_i\left(t_0\right) = c_i \left(i = 1, 2, ..., n\right)$, exist $\varepsilon > 0$, that for any solution $y\left(t\right)$ of (1.2) to satisfy the condition $\left\|y\left(t\right)\right\| < \varepsilon$. $y\left(t\right) = 0$ is **asymptotically stable** if it is stable and satisfy $\lim_{t\to\infty}\left\|y\left(t\right)\right\| = 0$.

Theorem 1.1. (Milici, et al., 2019; Petras, 2011). The equilibrium point $E^*\left(y_1^*, y_2^*, ..., y_n^*\right)$ of the system (1.2) is locally asymptotically stable if all the eigenvalues $\lambda_i \left(i = 1, 2, ..., n\right)$ of the Jacobian matrix $J = \frac{\partial f}{\partial y}$, $f = \left[f_1, f_2, ..., f_n\right]^T$ satisfy the condition: $\left|\arg\left(eig\left(J\right)\right)\right| = \left|\arg\left(\lambda_i\right)\right| > v\frac{\pi}{2}$, $i = 1, 2, ..., n.$

The equilibrium point is called as a non-hyperbolic if $\left|\arg\left(eig\left(J\right)\right)\right| = \left|\arg\left(\lambda_i\right)\right| \neq v\pi$

Definition 1.4. (Petras, 2011). For a three-dimensional system, the equilibrium point $E^*\left(y_1^*, y_2^*, y_3^*\right)$ is a

a. **Node:** when all the values $\lambda_i \left(i = 1, 2, 3, ..., n\right)$ of $J = \frac{\partial f}{\partial y}$ are real with the same sign.
b. **Saddle:** when all eigenvalues are real and at least one of them is positive and at least one is negative; Saddles are always unstable.
c. **Focus-Node** when it has one real eigenvalue and a pair of complex-conjugate eigenvalues, and all eigenvalues have real parts of the same sign; The equilibrium is stable (unstable) when the sign is negative (positive);
d. **Saddle-Focus:** when it has one real eigenvalue with the sign opposite to the sign of the real part of a pair of complex-conjugate eigenvalues; This type of equilibrium is always unstable.

Definition 1.5. (Petras, 2011). Lorenz system is described as:

$$\begin{aligned}
\overline{D}^{v}x\left(t\right) &= a\left(y\left(t\right) - x\left(t\right)\right) \\
\overline{D}^{v}y\left(t\right) &= x\left(t\right)\left(c - z\left(t\right)\right) + dy\left(t\right) \\
\overline{D}^{v}z\left(x\right) &= x\left(t\right)y\left(t\right) - bz\left(t\right)
\end{aligned} \tag{1.3}$$

$x(0)=1,\ y(0)=1,\ z(0)=1,\ \ a=10,\ b=8/3,\ c=24.74$, varied $d\in[-1,\ 10]$ with fractional order $v=0.998$, with equilibrium points of (1.3), $E_0=(0,0,0)$ the trivial one and $E_{1/2}=\left(\pm\sqrt{b(c+d)},\ \pm\sqrt{b(c+d)},\ c+d\right)$ and Jacobian matrix $J=\begin{bmatrix} -a & a & 0 \\ c-z^* & d & -x^* \\ y^* & x^* & -b \end{bmatrix}$.

2. Numerical Method

In recent years, the numerical approximation for the solutions of fractional order dynamical systems has attracted increasing attention in many fields of applied sciences and engineering. Many researchers considered the trapezoidal method, predictor-corrector method, extrapolation method, and spectral method. These methods are appropriate options if the resulting system of equations is linear but they present a high computational cost when the problem we are solving is badly conditioned or nonlinear. In this paper, the dynamics of a generalized form of the fractional-order Lorenz system are investigated by employing a modified version of the Runge-Kutta 4 method (RK4) (Lunch, 2007; Milici, et al., 2019). The method is very simple and very effective for solving differential equations of fractional order, it may be used.

The problem: (Miller, & Ross, 1993). A fractional differential equation for $v\in(0,1]$ and initial condition $y(t_0)=y_0,\ t\in[t_0,T]$,
${}_{t_0}\overline{D}_t^{v}\,y(t)=f(t,y(t))$. Where the $\overline{D}_t^{v}$ represent the Caputo fractional derivative.

2.1 Runge-Kutta 4 method (RK4)

Let $y(t)\in C[t_0,\ T],\ t_0$ is called the base point of fractional derivative. We set $t_n=t_0+nh,\ n=0,1,...,N^m$ where $h=(T-t_0)/N^m$ is the step size, *N* is a positive integer and $m\geq 1/v$. For the existence and uniqueness of the solution, we consider the Theorem 1.1. In the sequel, we assume $f(t,y)$ has continuous partial derivatives with respect to *t* and *y* to as high an order as we want.
The approximate solution can be obtained
from the expansion. Now, we introduce an *s*-stage fractional-order Runge–Kutta (FRK) method, which is discussed completely with s = 4 stage.

Definition 2.1. (Milici, et al., 2019). A family of *s*-stage FRK methods is defined as

$K_1=h^v f(t,y)$,

$K_2=h^v f(t+c_2h,\ y+a_{21}K_1)$

$K_3=h^v f(t+c_3h,\ y+a_{31}K_1+a_{32}K_2)$,

...

$K_s=h^v f(t+c_sh,\ y+a_{s1}K_1+a_{s2}K_2+...+a_{s,s-1}K_{s-1})$

where $y_{n+1} = y_n + \sum_{i=1}^{s} \omega_i K_i$, and the unknown coefficients $\{a_{ij}\}_{i=2,j=1}^{s,i-1}$ and the unknown weights $\{c_i\}_{i=2}^{s}$, $\{\omega_i\}_{i=1}^{s}$ has to be determined. For $s=4$ we use the following scheme:

$$y_{n+1} = y_n + \frac{h^\nu}{6\Gamma(\nu+1)}(K_1 + 2K_2 + 2K_3 + K_4)$$

where

$$K_1 = f(t_n, y_n)$$

$$K_2 = f\left(t_n + \frac{h^\nu}{\Gamma(2\nu+1)}, \right), y_n + \frac{1}{2}\frac{h^\nu}{\Gamma(2\nu+1)}$$

$$K_3 = f\left(t_n + \frac{1}{2}\frac{h^\nu}{\Gamma(2\nu+1)}, y_n + \frac{1}{2}\frac{h^\nu}{\Gamma(2\nu+1)}\right)$$

$$K_4 = f\left(t_n + \frac{h^\nu}{\Gamma(2\nu+1)}, y_n + \frac{h^\nu}{\Gamma(2\nu+1)}\right)$$

Caputo fractional derivatives of composite function $f(t, y(t))$ can be computed by fractional Taylor series.

3. Numerical Simulations

In this section we present simulation results of the modified Lorenz's system (1.3) on three dimensional plot, taking standard parameters (Rasimi, & Salihi, 2022) $a=10, b=8/3, c=24.74$, varied $d \in [-1, 10]$ with fractional order $\nu = 0.998$ and initial conditions $x(0)=1, y(0)=1, z(0)=1,$ to analyze its chaotic behavior using RK4 method. Modification of RK4 using its vectorial form, will be shown as special numerical method with intervention on standard above mention method. Starting with $d=-1$, the system (1.3) takes the standard form of Lorenz's system, its behavior is chaotic, shown in (Rasimi, & Salihi, 2022). In Figure 1, we plot the 3D behavior of (1.3) for $d=4.3179$, $d=4.65$ to see the change of its state, and then to divide the interval $d \in [-1, 10]$ by plots in Figure 2 for $d=6$, $d=9$.

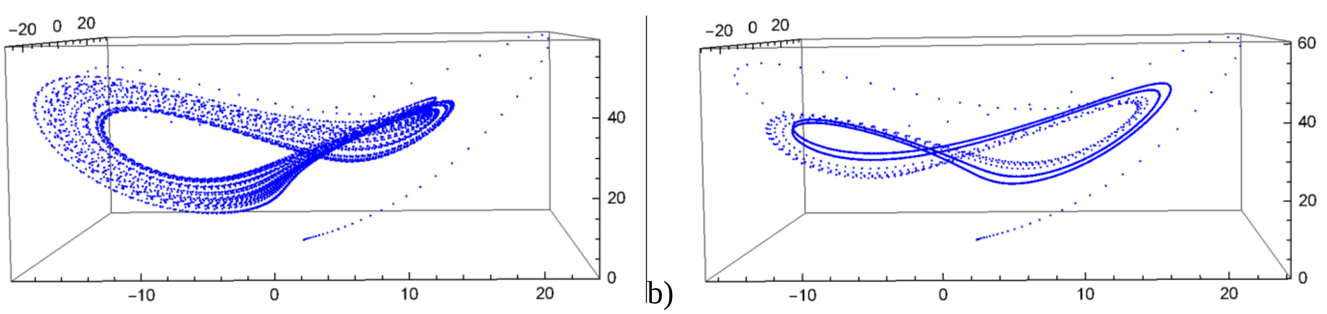
a)


Figure 1. 3D phase portrait of system (1.3) with α = 0.998, initial conditions $(x_0, y_0, z_0) = (1,1,1)$, step size h = 0.01, time t = 100s, parameters $a=10, b=8/3, c=24.74$ and **a)** $d=4.3179$ **b)** $d=4.65$ using Vectorial RK4

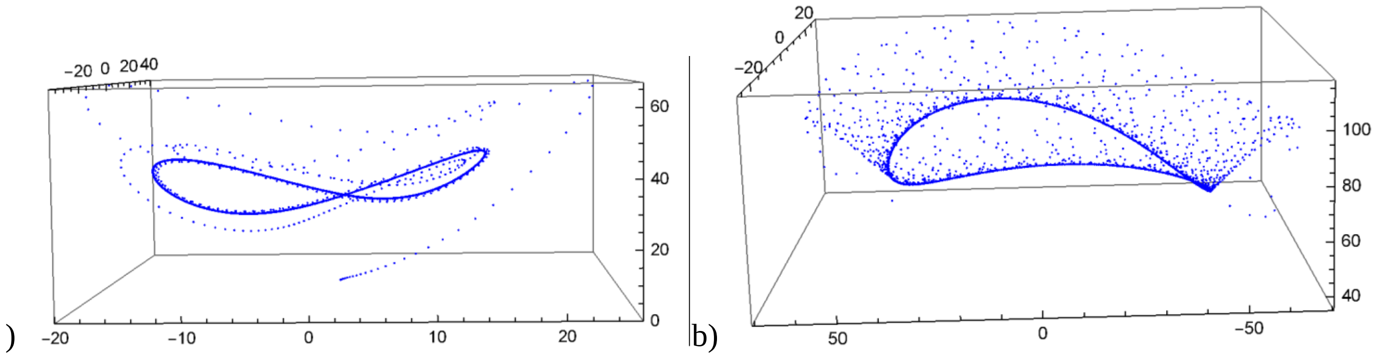


Figure 2. 3D phase portrait of system (1.3) with α = 0.998, initial conditions $(x_0, y_0, z_0) = (1,1,1)$, step size h = 0.01, time t = 100s, parameters $a = 10,\ b = 8/3,\ c = 24.74$ and **a)** $d = 6$ **b)** $d = 9$ using Vectorial RK4.

For fixed value of $d = 4.3179$, the initial values of Jacobian matrix for the first equilibrium point $E_0 = (0,0,0)$ are $(-20.1226,\ 14.4405,\ -2.66667)$, with $\left|\arg(\lambda_1)\right| = \pi > 0{,}998\frac{\pi}{2}$, for $E_{1/2} = \left(\pm\sqrt{77.4877},\ \pm\sqrt{77.4877},\ 29.0579\right)$ are $(-11.7231,\ 1.68715+11.3732i,\ 1.68715-11.3732i)$, with $\left|\arg(\lambda_1)\right| = 1.42353 < 0{,}998\frac{\pi}{2}$. $E_0 = (0,0,0)$ is asymptotically stable Saddle point, but from the results of Theorem 1.1 we can't say the same for $E_{1/2} = \left(\pm\sqrt{77.4877},\ \pm\sqrt{77.4877},\ 29.0579\right)$, they are Focus-Node stable points.

For fixed value of $d = 9$, the initial values of Jacobian matrix for the first equilibrium point $E_0 = (0,0,0)$ are $(-18.8753, 17.8753, -2.6667)$, with $\left|\arg(\lambda_1)\right| = \pi > 0{,}998\frac{\pi}{2}$, for $E_{1/2} = \left(\pm\sqrt{89.9733},\ \pm\sqrt{89.9733},\ 33.74\right)$ are $(-1.2215+13.2974i,\ -1.2215-13.2974i,\ -10.7919)$, with $\left|\arg(\lambda_1)\right| = 1.4792 < 0{,}998\frac{\pi}{2}$. $E_0 = (0,0,0)$ is asymptotically stable Saddle point, but from the results of Theorem 1.1 we can't say the same for $E_{1/2} = \left(\pm\sqrt{89.9733},\ \pm\sqrt{89.9733},\ 33.74\right)$, they are Focus-Focus unstable points.

With aim to divide the interval of varied $d \in [-1, 10]$, for $d \in [-1,\ 4]$ the system (1.3) is chaotic, according to many numerical experiments with RK4 and Vectorial RK4.

We can see changes in the state of the system (1.3) for $d = 4.3179$ when the system shows mostly periodic behavior. For $d \in [4.3179,\ 4.65]$ system becomes chaotic till $d \approx 9$ when it starts to has a non-chaotic behavior. The 3D phase portrait of the system according to varied d are presented in Figure 1 and Figure 2, results taken by numerical experiments using Vectorial RK4.

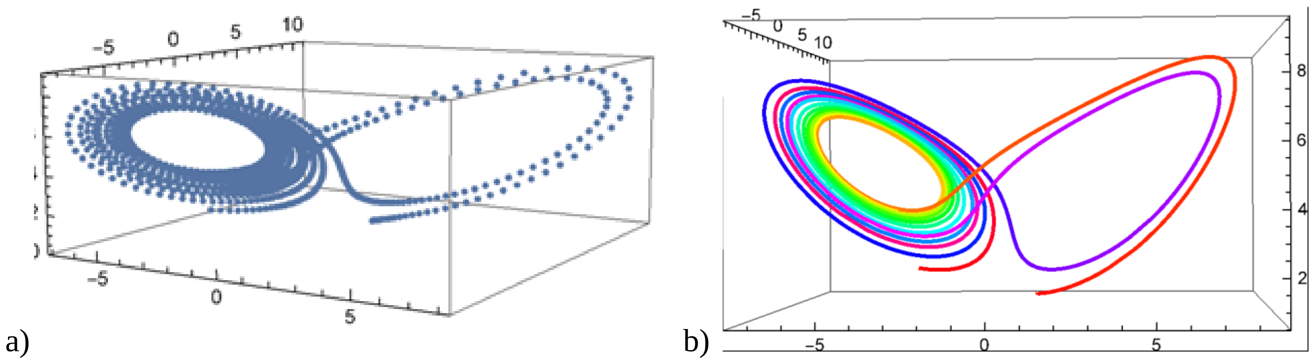


Figure 3. 3D phase portrait of system (1.3) with v= 0.998, initial conditions $(x_0, y_0, z_0) = (1,1,1)$, step size h = 0.01, time t = 100s, parameters $a = 10,\ b = 8/3,\ c = 24.74$ and $d = 4.3179$ using interpolation of RK4

In Figure 3. We plot 3D phase portrait of system (1.3) for parameters $a = 10,\ b = 8/3,\ c = 24.74$ and $d = 4.3179$ using RK4 and its interpolation. The results are almost the same with Figure 1 a), we can see clearly the mostly periodic behavior of system, with two different periods. We chose to explore numerically the system with Vectorial RK4, like a modification of RK4, and Interpolation of RK4, with the aim to be faster and more precisely in computations.

4. Conclusions

This present analysis exhibits the applicability of the numerical method RK4 and its modification using its vectorial interpretation to show chaotic behavior of the fractional order modified Lorenz system (1.3). The work emphasized our belief that the methods are reliable technique to handle nonlinear fractional differential systems. According to numerical experiments we divided the interval $d \in [-1, 10]$, to $d \in [-1,\ 4]$, $d \in [4.3179,\ 4.65]$ till $d \approx 9$, where the system (1.3) shows different behavior using Vectorial RK4.

Changes of the state of system (1.3) are in $d = 4.3179$ with periodic behavior, in $d \in [4.3179,\ 4.65]$ where it becomes chaotic and from $d \approx 9$ it starts with a non-chaotic behavior. The 3D phase portrait of the system according to varied d are presented in Figure 1 and Figure 2, with results taken by numerical experiments using Vectorial RK4. The method is fast, with timing 0.0625s of computation.

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