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Professional paper

SOME COMPUTATIONS OF NUMERICAL SEMIGROUPS WITH EMBEDDING DIMENSION 3 ON GAP

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Abstract

This paper aims to present some computations related to numerical semigroups with embedding dimension 3. We study a simple problem regarding the packaging methods of some products connected with some invariants of the numerical semigroup. We tried to define the algorithm that produces simultaneously a presentation of the semigroup by three generators and three relations and show how to use it for visualizing the different ways of buying certain products in the market We use the GAP package to find the exact number of products we want to buy, no one above no one below, where amounts correspond with finite sums of integers in the set $\{a, b, c\}$ or numerical semigroup $\langle a, b, c\rangle$.

Keywords: Numerical semigroups; Gap package; Minimal presentations; Diagram.

1. Introduction

Semigroups have found applications in different areas. [5] deals with the applications of semigroups in general and regular semigroups in particular. There are studied semigroup applications to computer science, biological science, and sociology. It discussed that semigroups can be used in biology to describe certain aspects in the crossing of organisms, in genetics, and in consideration of metabolisms. Also [6] presents a panorama of various applications: Semigroups and asymptotics, Semigroups for dynamical systems, Form methods and perturbations of semigroups, Semigroups, and partial differential equations.
The following definitions are taken from [1] and [4].

Definition 1.1 Let S be a numerical semigroup and A be a subset of S. We say that A is a system of generators of S if $S = \{k_1a_1 + \cdots + k_na_n \mid n, k_1,..., k_n \in N, a_1,..., a_n \in A\}$. The set A is a minimal system of generators of S if no proper subset of A is a system of generators of S.

Definition 1.2 For the nonempty subset S of $\mathbb{N}_{\mathbf{0}}$, we will say that it is a numerical semigroup if S is closed concerning to the addition operation, it contains 0 and $\mathbb{N}_{\mathbf{0}}\backslash S$ is a finite set.

Definition 1.3 *Let* S *be a numerical semigroup and let* $\{\mathrm{n}_1 < \mathrm{n}_2 < \cdots < \mathrm{n}_\mathrm{p}\}$ *be its minimal system of generators. Then*

- n_1 *is known as the* **multiplicit**y *of* S, *denoted by* $m(S)$. *It is the smallest element in the minimal generator set of* S.
- *The* ***cardinality*** *of the minimal system of generators,* p, *is called the* embedding dimension *of* S *and will be denoted by* $e(S)$.

- *The* ***Apery set*** n *in* S, where $n \neq 0$ *is* $Ap(S, n) = \{s \in S \mid s - n \notin S\}$.
- The set of elements in $G(S) = N \backslash S$ is known as the set of **gaps** of S. Its cardinality is the **genus** of S, $g(S)$.
- A gap g of a numerical semigroup S is an isolated gap if $g - 1, g + 1 \in S$. We will denote by $I(S)$ the set of all **isolated gaps** of S.
- **Frobenius number** of S is the greatest integer not in S. In the literature it is sometimes replaced by the conductor of S, which is the least integer x such that $x + n \in S$ for all $n \in N$.
- The smallest element of S such that all larger integers are likewise elements of S is called the **conductor**, it is F(S) + 1.
- An integer z is a **pseudo-Frobenius** number of S if $z + S \setminus \{0\} \subseteq S$. Thus the Frobenius number of S is one of its pseudo-Frobenius numbers. The **type** of a numerical semigroup is the cardinality of the set of its pseudo-Frobenius numbers.

2. Main part

Next, find some invariants of numerical semigroup $S = \langle 5,7,23 \rangle$ with embedding dimension 3 on GAP.
First, we loaded the numericalsgps package and then execute the code for some invariants of S as following:

```
gap> LoadPackage("numericalsgps");;
gap> S:=NumericalSemigroup(5,7,23);
<Numerical semigroup with 3 generators>
gap> EmbeddingDimension(S); 3
gap> S{[50..70]};
[ 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80 ]
gap> NextElementOfNumericalSemigroup(16,S); 17
gap> SmallElementsOfNumericalSemigroup(S); [ 0, 5, 7, 10, 12, 14, 15, 17, 19 ]
gap> MinimalPresentationOfNumericalSemigroup(S);
[ [ [ 0, 4, 0 ], [ 1, 0, 1 ] ], [ [ 5, 3, 0 ], [ 0, 0, 2 ] ], [ [ 6, 0, 0 ]
gap> AperyListOfNumericalSemigroupWRTElement(S,5); [ 0, 21, 7, 23, 14 ]
gap> AperyListOfNumericalSemigroup(S); [ 0, 21, 7, 23, 14 ]
gap> FrobeniusNumber(S);18
gap> ConductorOfNumericalSemigroup(S);19
gap> GenusOfNumericalSemigroup(S);11
gap> MultiplicityOfNumericalSemigroup(S);5
gap> PseudoFrobeniusOfNumericalSemigroup(S);[ 16, 18 ]
```

This means that semigroup S is { 0, 5, 7, 10, 12, 14, 15, 17, 19, →}, where the arrow means that every integer larger than 19 is in the set. NextElementOfNumericalSemigroup(S, r) returns the least integer greater than r belonging to S. If we take a nonzero element n in the semigroup, its Apéry set has exactly n elements.

Lemma 2.1. [1] Let A be a nonempty subset of **N**. Then $\langle A \rangle$ is *a numerical semigroup if and only if* $gcd(A) = 1$.

Lemma 2.2. [2] *Let* S *be a numerical semigroup and let* n *be a nonzero element of* S. *Then* $Ap(S,n) = \{0 = w(0), w(1), \ldots, w(n-1)\}$, *where* $w(i)$ *is the least element of* S *congruent with* i *modulo* n, *for all* $i \in \{0, \ldots, n-1\}$.

Next, we prove a very important property of Apery set of numerical semigroups with embedding dimension three.

Lemma 2.3. *Let* S *be a numerical semigroup with embedding dimension three generated by* $\{n_1, n_2, \ldots, n_p\}$. *Let* $d = gcd\{n_1, n_2, \ldots, n_{p-1}\}$ *and set* $T = \langle n_1/d, n_2/d, \ldots, n_{p-1}/d, n_p \rangle$. *Then*

$$Ap(S, n_p) = d(Ap(T, n_p)).$$

Proof.
If $w \in Ap(S, n_p)$, then $w \in \langle n_1, \ldots, n_{p-1} \rangle$. Hence $w/d \in \langle n_1/d, \ldots, n_{p-1}/d \rangle \subseteq T$. If $w/d - np \in T$, then $w - dnp \in S$, which is impossible. Now take $w \in Ap(T, np)$. Then $w \in \langle n_1/d, \ldots, n_{p-1}/d \rangle$, and thus $dw \in \langle n_1, \ldots, n_{p-1} \rangle \subseteq S$. If $dw - n_p$ also belongs to S, then $dw - n_p = \lambda_1 n_1 + \cdots + \lambda_{p-1} n_{p-1} + \lambda_p n_p$ for some $\lambda_1, \ldots, \lambda_p \in N$. Since S is a numerical semigroup $gcd\{n_1, \ldots, n_p\} = 1$, which implies that $gcd\{d, n_p\} = 1$. This leads to $d | (\lambda_p + 1)$, because $(\lambda_p + 1)n_p = dw - (\lambda_1 n_1 + \cdots + \lambda_{p-1} n_{p-1})$. But then $w = \frac{\lambda_1 n_1}{d} + \cdots + \frac{\lambda_{p-1} n_{p-1}}{d} + \frac{\lambda_p + 1}{d} n_p$, with $(\lambda_p + 1)/d$ a positive integer, contradicting that $w \in Ap(T, n_p)$.

2.1. Application of numerical semigroup in the market: The problem is simple. We want to buy a certain amount of any product and the store offers boxes of a, b, c (as it was in MacDonnald's sometime in the last century). We want to have the exact number of products we want to buy, no one above no one below. Notice that the amounts correspond with finite sums of integers in the set $\{a, b, c\}$ and thus we are talking about the numerical semigroup $\langle a, b, c \rangle$. So to show how many ways of buying these products do we have in the range {0,…,60} we use gap packet numericalsgps and gap function NrRestrictedPartitions as follows:

S:=NumericalSemigroup(a, b, c);

```
Plot([0..60], x->NrRestrictedPartitions(x,[ a,b,c]),
 rec(
title := "Different ways to buy this many products",
         xaxis := "n",
yaxis := "Number of expressions of n in terms of the boxes",
xaxis := "Number of products, n", type:="bar"));
```

Example 1.
How many ways of buying products do we have in the range {0,…,60} if the buy offers boxes of 6,13,23 amounts?

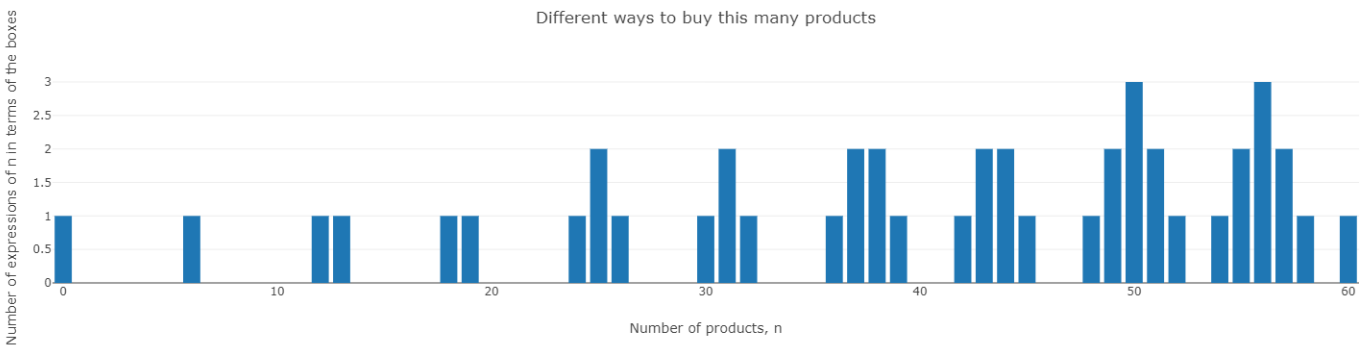

Figure 1.

It is clear that for elements outside ⟨6,13,25⟩ there will be no possible choice of boxes to get that specific number of products.

Gaps(S);
[1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 13, 14, 16, 17, 19, 20, 22, 23, 26, 28, 29, 32, 34, 35, 38, 41, 44, 47, 53, 59].

So the amounts of products that we cannot buy with this box offer are precisely the gaps of S.
Next to the above bar diagram we give the minimum number of boxes required to obtain the dessired number of products in the interval $[0..60]$.

```
sizes:=[ a,b,c];;
realizable:=Intersection(S,[0..60]);;
Plot([realizable, x->NrRestrictedPartitions(x,sizes),
 rec(title := "Different ways to buy this many products",
         xaxis := "n",
yaxis := "number of expressions of n in terms of the boxes",
xaxis := "number of products, n", type:="bar")],
[realizable, x->Minimum(List(RestrictedPartitions(x,sizes),Length)),
 rec(type:="line", name:="min # boxes",)],
```

```
[realizable, x->Maximum(List(RestrictedPartitions(x,sizes),Length)),
 rec(type:="line", name:="max # boxes",)]);
```

Example 2.

How many ways of buying minimum boxes of products do we have in the range $\{0, \dots, 60\}$ if the store offers boxes 6, 13 and 25 and we want an exact number of products?

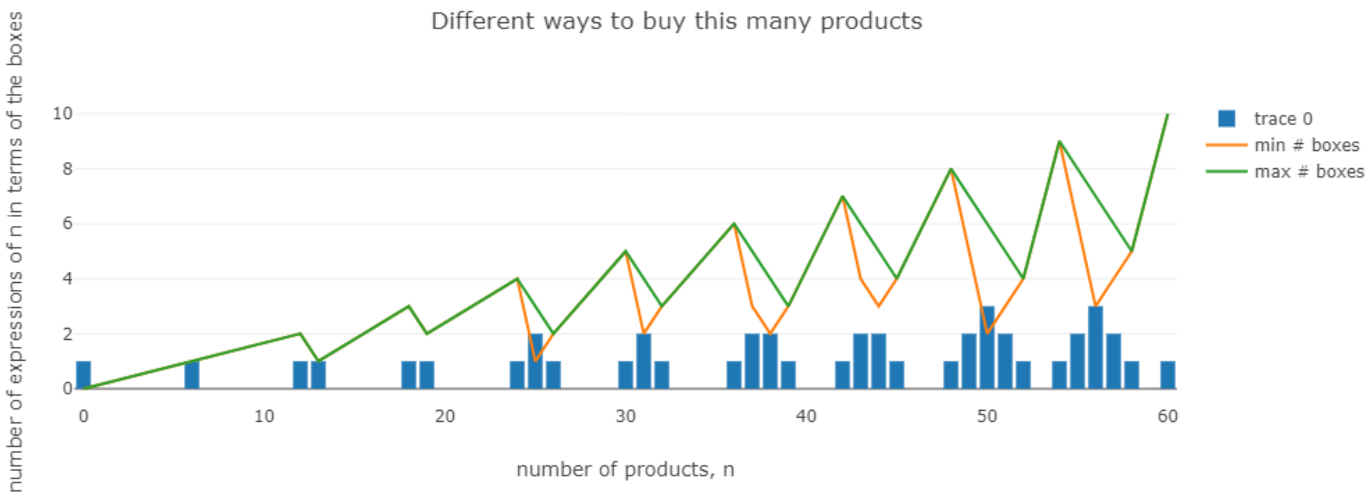

Figure 2.

From diagram we see that to buy exactly 50 products we have possibilities: Two boxes with 25 products (minimal number of boxes) or four boxes with 6 products and two with 13 products (maximal number of boxes). Also, if we put

```
gap> p:=RestrictedPartitions(50,sizes);
[ [ 13, 13, 6, 6, 6, 6], [ 25, 13, 6, 6], [ 25, 25]]
```

we get the same answer.

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