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Convergence in cone metric spaces with normal cones

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Abstract

There have been a lot of successful attempts to generalize the notion of metric space. If the set of real numbers as an image of the distance function is replaced by a Banach space ordered by a (solid) cone, then the cone metric space is defined. It is known that a cone defined in the Banach space is either normal or non-normal. There is a definition for convergence of sequences in cone metric spaces. This paper was focused on seeing some properties of convergence of sequences in cone metric spaces where the cone with respect to which the order has been defined is normal. Cauchy sequences and complete cone metric spaces are defined. Some examples are provided as well as the Banach Contraction Principle in a cone metric space.

Keywords: Convergence, Cone Metric Space, Normal Cones.

1. Introduction

One of the most important concepts in vector spaces is the concept of cones. It is used mostly to provide an order on vector spaces. A lot of results about cones can be found in [1].

Even though a cone can be defined in any vector space, in this paper we need a definition of cones in real Banach spaces.

Definition 1.1: Let E be a real Banach space and P a nonempty subset of E such that:

- P is closed and $P \neq \{0\}$
- $\forall \alpha, \beta \geq 0$ and $\forall x, y \in P$, $\alpha x + \beta y \in P$
- $x \in P$ and $-x \in P$ imply $x = 0$.

Then P is said to be a *cone* in the real Banach space E.

Example 1.1: Some examples of cones are:

- $[0, \infty)$ is a cone in $\mathbb{R}$ with the Euclidean norm.
- $P = \{(x, y) \mid x \geq 0, y \geq 0\}$ is a cone in $\mathbb{R}^2$ with the Euclidean norm.
- $\{x \in C[0,1] \mid x(t) \geq 0,\ \forall t \in [0,1]\}$ is a cone in $C[0,1]$ with the supremum norm.
- $\{x \in C[0,1] \mid x(t) \geq 0,\ \forall t \in [0,1]\}$ is a cone in $C[0,1]$ with the norm defined by $\|x\| = \|x\|_\infty + \|x'\|_\infty$.

Given a cone P in a real Banach space E we define a partial order on E by $x \preceq y$ if $y - x \in P$. The relation "$\preceq$" is a partial order on E since it is

- Reflexive: $\forall x \in E$, $x \preceq x$ since $x - x = 0 \in P$
- Antisymmetric: $\forall x, y \in E$, $x \preceq y$ and $y \preceq x$ iff $x = y$ since $y - x \in P$ and $x - y \in P$ imply $x - y = 0$ by definition
- Transitive: $\forall x, y, z \in E$ if $x \preceq y$, $y \preceq z$ then $x \preceq z$ since $y - x \in P$ and $z - y \in P$ imply $z - x = (z - y) + (y - x) \in P$.

In definition 1.1 it is given the general definition of cones, but depending on different properties we can divide cones in different types. Next we introduce some of them:

Definition 1.2: Let P be a cone in the real Banach space. P is said to be

- *Solid* if $\operatorname{int} P \neq \varnothing$
- *Normal* if there exists a number $K > 0$ such that $\forall x, y \in E,\ 0 \preceq x \preceq y$ imply $\|x\| \leq K\|y\|$. The least number *K* satisfying this is called the normal constant of *P*.
- *Monotone* if $\forall x, y \in E,\ 0 \preceq x \preceq y$ imply $\|x\| \leq \|y\|$
- *Regular* if every increasing sequence which is bounded from above is convergent

It is easy to notice that any monotone cone is normal (with the constant 1).

'Every decreasing sequence bounded from below is convergent' is an equivalent statement defining regularity of a cone. In [5] it is proven that the regularity of the cone implies its normality.

Next theorem is proved in [4].

Theorem 1.1: The following conditions are equivalent:

- *P* is a normal cone
- For arbitrary sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in *E* such that $x_n \preceq y_n \preceq z_n,\ \forall n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} z_n = x$ imply $\lim\limits_{n\to\infty} y_n = x$
- There exists a norm $\|\cdot\|_1$ on E, equivalent to the given norm $\|\cdot\|$, such that the cone *P* is monotone w.r.t. $\|\cdot\|_1$

While first three cones in Example 1.1 are normal, the forth one is not. Indeed, let $P = \{x \in C[0,1] \mid x(t) \geq 0,\ \forall t \in [0,1]\}$ be a cone in $C[0,1]$ with the norm defined by $\|x\| = \|x\|_\infty + \|x'\|_\infty$. Consider the sequences $x_n(t) = \dfrac{t^n}{n}$ and $y_n(t) = \dfrac{1}{n}$.

There holds $0 \preceq x_n \preceq y_n$ and $\lim\limits_{n\to\infty}\|y_n\| = 0$ but $\lim\limits_{n\to\infty}\|x_n\| = \lim\limits_{n\to\infty}\left\|\dfrac{t^n}{n}\right\|_\infty + \lim\limits_{n\to\infty}\|t^{n-1}\|_\infty = 0 + 1 = 1$ which contradicts the sandwich theorem (Theorem 1.1) i.e. *P* is a non-normal cone. This fact is proven in [5] in a different way.

2. Cone metric spaces

In 2017 in [3] it was introduced the concept of cone metric space as a generalization of metric spaces.

Let *X* be a nonempty set and $(E, \preceq)$ a real Banach space equipped by an order with respect to a solid cone *P*.

Definition 2.1: The function $d : X \times X \to E$ defined so that:

- $d(x, y) \succeq 0,\ \forall x, y \in X;$
- $d(x, y) = 0$ iff $x = y;$
- $d(x, y) = d(y, x),\ \forall x, y \in X;$
- $d(x, y) \preceq d(x, z) + d(z, y),\ \forall x, y, z \in X$

is called *cone metrics*, and the pair *(X,d)* is a *cone metric space*.

Here are some examples of cone metric spaces given in [2]:

Example 2.1: Let $E = \mathbb{R}^2$, $P = \{(x, y) \mid x, y \geq 0\}$, $X = \mathbb{R}$ and $d : X \times X \to E$ defined with $d(x, y) = (|x - y|, \alpha |x - y|)$ where $\alpha \geq 0$.

Example 2.2: Let $E = \mathbb{R}^n$, $P = \{(x_1, x_2, ..., x_n) \mid x_i \geq 0,\ \forall i \leq n\}$, $X = \mathbb{R}$ and $d : X \times X \to E$ defined with $d(x, y) = (|x - y|, \alpha_1 |x - y|, \alpha_2 |x - y|, ..., \alpha_{n-1} |x - y|)$ where $\alpha_i \geq 0,\ \forall i \leq n-1$.

Example 2.3: Let $E = \left(C[0, \infty), \|\cdot\|_\infty\right)$, $P = \{x \in E \mid x(t) \geq 0,\ \forall t \in [0, \infty)\}$, (X, ρ) a metric space and $d : X \times X \to E$ defined with $d(x, y) = \rho(x, y)\varphi$ where $\varphi : [0,1] \to \mathbb{R}^+$ is a continuous function.

Example 2.4: Let $q > 0$, $E = l^q$, $P = \{\{x_n\} \in E \mid x_n \geq 0,\ \forall n \in \mathbb{N}\}$, (X, ρ) a metric space and $d : X \times X \to E$ defined with $d(x, y) = \left\{\sqrt[q]{\dfrac{\rho(x, y)}{2^n}}\right\}_{n \in \mathbb{N}}$.

In each of upper examples (X,d) is a cone metric space.

3. Convergence in cone metric spaces

In [3] it is defined the convergence in cone metric spaces. Before we give that definition, we need to introduce some notations.

Let *E* be a real Banach space and *P* a solid cone. We denote $x \prec\prec y$ when $y - x \in IntP,$ and $x \succ\succ y$ when $y \prec\prec x,\ \forall x, y \in X.$

Definition 3.1: Let (X,d) be a cone metric space, $\{x_n\}$ a sequence in X and $x \in X.$ If $\forall c \succ\succ 0, \exists n_0(c)$ such that $\forall n > n_0$ we have $d(x, x_n) \prec\prec c,$ then we say that $\{x_n\}$ *converges* to x as $n \to \infty$ and we denote it with $\lim_{n\to\infty} x_n = x.$

Cauchy sequences are defined just as in metric spaces.

Definition 3.2: If $\forall c \succ\succ 0, \exists n_0(c)$ such that $\forall m, n > n_0$ we have $d(x_m, x_n) \prec\prec c,$ then we say that $\{x_n\}$ is a *Cauchy sequence*.

Consequently, every convergent sequence is a Cauchy sequence and if every Cauchy sequence in *X* is convergent in *X*, then *X* is called a *complete cone metric space*.

If the cone is a solid normal cone, some results regarding convergence occur. Most of the results in the next theorem are given in [3], but here we prove them in a different way.

Theorem 3.1: Let *E* be a real Banach space and *P* a normal and solid cone. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in *X*. Then the following claims hold:

i) $\{x_n\}$ converges to *x* in *X* iff $\lim_{n\to\infty} d(x_n, x) = 0.$

ii) If $\{x_n\}$ converges to *x* and $\{x_n\}$ converges to *y* in *X*, then *x*=*y*.

iii) If $\{x_n\}$ converges to *x* in *X* and $\{y_n\}$ converges to *y* in *X*, then $\lim_{n\to\infty} d(x_n, y_n) = d(x, y).$

iv) $\{x_n\}$ is a Cauchy sequence iff $\lim_{n,m\to\infty} d(x_n, x_m) = 0.$

Proof: i) See [3]

ii) Let $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} x_n = y.$ By i) we have that $\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x_n, y) = 0.$ Then $d(x,y) \preceq \lim_{n\to\infty} d(x, x_n) + \lim_{n\to\infty} d(x_n, y) = 0$ hence *x*=*y*.

iii) Let $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y.$ By i) we have that

$\lim_{n\to\infty} d(x_n, y_n) \preceq \lim_{n\to\infty} d(x_n, x) + \lim_{n\to\infty} d(x, y) + \lim_{n\to\infty} d(y, y_n) = d(x, y).$ On the other hand

$d(x,y) \preceq d(x, x_n) + d(x_n, y_n) + d(y_n, y), \forall n \in \mathbb{N}$, hence $d(x,y) \preceq \lim_{n\to\infty} d(x, x_n) + \lim_{n\to\infty} d(x_n, y_n) + \lim_{n\to\infty} d(y_n, y) =$

$= \lim_{n\to\infty} d(x_n, y_n).$ Then $d(x, y) = \lim_{n\to\infty} d(x_n, y_n).$

iv) The sequence $\{x_n\}$ is a Cauchy sequence iff $\forall c \succ\succ 0, \exists n_0(c)$ such that $\forall m, n > n_0$ we have $d(x_n, x_m) \prec\prec c$ i.e. $\lim_{n,m\to 0} d(x_n, x_m) = 0.$

As an application to the previous results we give the Banach Contraction Principle version in cone metric space with normal cones.

Theorem 3.2: [3] Let (X,d) be a complete cone metric space, *P* be a normal cone with normal constant *K*, $T : X \to X$ a function and $k \in [0,1)$ such that

$$d(T(x), T(y)) \preceq kd(x, y),\ \forall x, y \in X,$$

Then *T* has a unique fixed point, and for any $x \in X$ the iterative sequence $\{T^n(x)\}$ converges to the fixed point.

Proof: Choose $x_0 \in X.$ Set $x_1 = T(x_0),\ \ x_2 = T(x_1) = T^2(x_0),\ \ldots,\ x_{n+1} = T(x_n) = T^{n+1}(x_0),\ \ldots$

We have $d(x_{n+1}, x_n) = d(T(x_n), T(x_{n-1})) = kd(x_n, x_{n-1}) = k^2 d(x_{n-1}, x_{n-2}) = \ldots = k^n d(x_1, x_0).$

So, for $n > m$ we have

$$d(x_n, x_m) \preceq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \ldots + d(x_{m+1}, x_m) \preceq (k^{n-1} + k^{n-2} + \ldots k^m) d(x_1, x_0) \preceq \frac{k^m}{1-k} d(x_1, x_0).$$

By the assumption that the cone is normal we get $\|d(x_n, x_m)\| \leq \frac{k^m}{1-k} K \|d(x_1, x_0)\|$ where K is the normal constant of the cone. This implies that $\lim_{n,m\to\infty} d(x_n, x_m) = 0,$ hence $\{x_n\}$ is a Cauchy sequence, and by the completeness of X, there exists an $x^* \in X$ s.t. $\{x_n\}$ converges to x^* in X.

x^* is a fixed point for T since $d(T(x^*), x^*) \preceq d(T(x^*), T(x_n)) + d(T(x_n), x^*) \preceq kd(x^*, x_n) + d(x_{n+1}, x^*) \to 0.$

x^* is unique since for any other fixed point y^* we get $d(x^*, y^*) = d(T(x^*), T(y^*)) \preceq kd(x^*, y^*)$ hence $d(x^*, y^*) = 0$ therefore $x^* = y^*$.

4. Conclusions

Cone metric space is a useful tool especially in fixed point theory since it enlarges the set of functions which can be proved to have a fixed point. Even though in some recent papers it is proved that most of the fixed point theorems in cone metric spaces are redundant, since they are a direct consequences of similar theorems in metric spaces, yet this is not a general case.

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