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Some results concerning the analytic representation of distributions

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Abstract

One property of distributions which is essentially different from the situation pertaining to locally integrating function is that distributions are infinitely differentiable. Every locally integrating function has a distributional derivative since it can be identified with a certain distribution. In contrast of classical analysis, a convergent sequence of distributions can always be differentiated and the resulting sequence converges to the derivative of the limit. The theory of Cauchy integral of distributions is motivated by the classical theory of the Cauchy integral. In the theory of distributional behaviour of analytic functions the following two topics are central: the representation of distributions in terms of boundary values of analytic functions and the representation of analytic functions in terms of distributions. In this paper we obtain some results related to boundary values of analytic representations of a sequence of distributions using Cauchy representation for distributions and analytic representations of distributions in different spaces. Arbitrary continuous complex valued function on real line cannot be analytically continued into the complex plane. It is possible to find a complex function which is analytic in a subset of complex plane and which represents the function by a jump arbitrarily close to the real axis.

Keywords: distributions, analytic representation, Cauchy representation, boundary values.

1. Introduction

With $C^{\infty}(\mathrm{R}^n)$ is denoted the space of all infinitely differentiable functions on $\mathbb{R}^n$ and $C_0^{\infty}(\mathbb{R}^n)$ denotes the subspace of $C^{\infty}(\mathbb{R}^n)$ that consists of those functions of $C^{\infty}(\mathbb{R}^n)$ which have compact support.

Definition 1.1. The support of f is the closure of the set $x \in \Omega$, of points for which f is different from zero $(f(x) \neq 0)$, and is denoted by supp f.

With D we denote the space of $C_0^{\infty}(\mathbb{R}^n)$ functions, called the set of test functions in which convergence is defined in the following way : a sequence $\{\varphi_\nu\}$ of functions $\varphi_\nu \in D$ converges to $\varphi \in D$ in D as $\nu \to \nu_0$ if and only if there is a compact set $K \subset \mathbb{R}^n$ such that supp$(\varphi_\nu) \subseteq K$ for each ν, supp$(\nu) \subseteq K$ and for every n-tiple k of nonnegative integers the sequence $\{f^{(k)}\varphi_\nu(t)\}$ converges to $f^{(k)}\varphi(t)$ uniformly on K as $\nu \to \nu_0$.

Distributions (or generalized functions) are objects that generalize the classical notion of functions in mathematical analysis. Distributions make it possible to differentiate functions whose derivatives do not exist in the classical sense.

Definition 1.2. A distribution T is continuous linear functional on D. Instead of writing $T(\varphi)$, it is conventional to write $<T,\varphi>$ for the value of T acting on a test function φ. The space of all distributions is denoted by D'.

Let φ be an element of one of the above function spaces D or S, and f be a function for which

$$< T_f, \varphi >= \int_{\mathbb{R}^n} f(t)\varphi(t)dt, \quad \varphi \in D(\varphi \in S)$$

exists and is finite. Then T_f is regular distribution on D (or S) generated by f.

The Heaviside function is defined by

$$H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

A very familiar toll of applied mathematics is the so-called Dirac delta function $\delta(x)$ which is usually defined by

$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases}$$

with

$$\int_{\mathbb{R}} \delta(x)dx = 1$$

and if φ is a continuous function on $\mathbb{R}$ then $\int_{\mathbb{R}} \delta(x)\varphi(x)dx = \varphi(0)$.

Definition 1. 3. We say that a function $O_\alpha = O_\alpha(\mathbb{R}^n)$ if φ is infinitely differentiable and if for each n-tuple β of nonnegative integers there exists a constant M_β such that $\left|D^\beta \varphi(t)\right| \leq M_\beta (1+|t|)^\alpha$, $t \in \mathbb{R}^n$

Converegence in the vector space O_α is defined as follows:

a sequence $\{\varphi_\lambda\}$ converges to φ in O_α as $\lambda \to \lambda_0$ if

1) each $\varphi_\lambda \in O_\alpha$,
2) for each β the sequence $\{D^\beta \varphi_\lambda\}$ converges uniformly on every compact subset of $\mathbb{R}^n$ to $D^\beta \varphi$ as $\lambda \to \lambda_0$,
3) for each β there exists a constant M_β, which is idependent of λ, such that $\left|D^\beta \varphi(t)\right| \leq M_\beta (1+|t|)^\alpha$, $t \in \mathbb{R}^n$.

The vector space O_α' is the space of all continuous linear functionals on O_α with continuity having the usual meaning that $< U, \varphi_\lambda > \to < U, \varphi >$ if $\varphi_\lambda \to \varphi$ in O_α for $U \in O_\alpha'$.

As in the space of D' we define the convergence in O_α' as : the sequence $\{U_\lambda\}$ of distributions in O_α' converges to U if $< U_\lambda, \varphi > \to < U, \varphi >$ for every $\varphi \in O_\alpha$ as $\lambda \to \lambda_0$.

An arbitrary continuous complex valued function on $\mathbb{R}$ cannot be analytically continued into the complex plane $\mathbb{C}$. In fact let $f(x)$, which maps $\mathbb{R}$ to $\mathbb{C}$, be a continuous function with compact support, and assume that a function $h(z)$ is its analytically continuation into □ . We have $f(x)=h(z),\ x\in$□ . This implies $h(x)=0$ on □ $-\operatorname{supp} f$, and by the uniqueness theorem we have $h(z)=0, z\in$□ .

Although it is impossible to represent an arbitrary $f(x)$ as the restriction of an analytic function, it is possible to find a function $f(z)$ which is analytic in a subset of □ and which represent $f(x)$ by a jump

$f(x+i\varepsilon)-f(x-i\varepsilon)$ arbitrarily close to real axis. This property is given in the following theorem.

Theorem 1.1. Let $f(t)$ map □ to □ and be a continuous function with $f(t)=O(1/\left|t\right|^{\alpha})$ for some $\alpha>0$ as $\left|t\right|\to\infty$. Let

$$\hat{f}(z)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(t)}{t-z}dt\ ,\quad z\in\Delta=\left\{\operatorname{Im} z\neq 0\right\}.$$

Then

$$\lim_{\varepsilon\to 0^{+}}(\hat{f}(x+i\varepsilon)-\hat{f}(x-i\varepsilon))=f(x)$$

uniformly on every compact subset of □ .

$\hat{f}(z)$ is known as Cauchy integral (representation) of f and the limit as analytic representation of f .

2 . Main Results

Theorem 2.1. Let $T\in O_{\alpha}^{'},\ \alpha\geq -1.$ Let

$$\hat{T}(z)=\frac{1}{2\pi i}<T,\frac{1}{t-z}>.$$

Then $\hat{T}(z)$ is an analytic function of z in the complement of the support of T .

$\hat{T}(z)$ represent the distribution T in the following sense:

$$\lim_{\varepsilon\to 0^{+}}\int_{-\infty}^{\infty}[\hat{T}(x+i\varepsilon)-\hat{T}(x-i\varepsilon)]\phi(x)dx=<T,\phi>\ \text{ for all } \phi\in D.$$

Proof. For $Imz\neq 0$, we have $\frac{1}{t-z}=0(\left|t\right|^{-1})$ and $D^{k}(t-z)^{-1}=[(-1)^{k}k!](t-z)^{-k-1}=0(\left|t\right|^{-k-1}), k=0,1,2,\ldots$

Hence $(t-z)^{-1}\in(O_{-1})$, and $(t-z)^{-1}\in O_{\alpha}$ for $\alpha\geq -1$. Therefore $\hat{T}(z)$ exist for $Imz\neq 0$. We observe that $\frac{1}{h}(\frac{1}{t-z-h}-\frac{1}{t-z})\to\frac{1}{(t-z)^{2}}$ in the sense of O_{-1}. Hence $\hat{T}(z)$ is analytic for $Imz\neq 0$. To show T that $\hat{T}(z)$ is analytic on the real axis in the complement of T , we may multiply with C^{∞} - function that is identically 1 on the neighbourhood of support of T and whose support approximates that of T . Hence $\hat{T}(z)$ is analytic as stated.

We assumed that $\phi\in D$, hence the integral $\int_{-\infty}^{\infty}[\hat{T}(x+i\varepsilon)-\hat{T}(x-i\varepsilon)]\phi(x)dx$ exist.

We can approximate he integral by Riemann sums and exchange summation and application of T .

Hence $O'_\alpha \subset O'_{-1}$ for $\alpha \geq -1$. Hence the previous theorem gives us a representation for all O'_α with $\alpha \geq -1$.

Theorem 2.2. Let $\{f_n(t)\}$ be a sequence of functions in $L^2(\mathbb{R})$ that converges to $f(t)$ in $L^2(\mathbb{R})$ as n tends to ∞ and let

$$\hat{f}_n(z) = \frac{1}{2\pi i} < f_n(t), \frac{1}{t-z} >, \qquad Imz \neq 0.$$

The sequence $\{\hat{f}_n(z)\}$ uniformly converges on every compact subset of $\mathbb{C}\backslash\mathbb{R}$ to the function $\hat{f}(z)$, where

$$\hat{f}(z) = \frac{1}{2\pi i} < f(t), \frac{1}{t-z} >, \qquad Imz \neq 0.$$

Proof. Let $z = x + iy$ be a complex number such that $Imz \neq 0$. Using the definitions of $\hat{f}_n(z)$ and $\hat{f}(z)$, we get that

$$\hat{f}_n(z) - \hat{f}(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f_n(t)}{t-z} dt - \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(t)}{t-z} dt =$$
$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f_n(t) - f(t)}{t-z} dt.$$

Since $f_n, f \in L^2(\mathbb{R})$ for every $n \in \mathbb{N}$ and $\frac{1}{t-z} \in L^2(\mathbb{R})$, for $Imz \neq 0$, we may apply the Schwarz's inequality. So, the following inequality

$$\left|\hat{f}_n(z) - \hat{f}(z)\right| \leq$$
$$\frac{1}{2\pi}\left[\int_{-\infty}^{\infty} |f_n(t) - f(t)|^2 dt\right]^{1/2} \left[\int_{-\infty}^{\infty} \frac{1}{|t-z|^2} dt\right]^{1/2}$$

holds.

By the assumption, the sequence $\{f_n(t)\}$ converges to the function $f(t)$ in $L^2(\mathbb{R})$ as n tends to infinity. So, for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have

$$\left[\int_{-\infty}^{\infty} |f_n(t) - f(t)|^2 dt\right]^{1/2} < \varepsilon.$$

We know that the quality

$$\left[\int_{-\infty}^{\infty} \frac{1}{|t-z|^2} dt\right]^{1/2}$$

is bounded.

Finally, for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $z \in \mathbb{C}$, for which $Imz \neq 0$, the inequality

$$\left|\hat{f}_n(z) - \hat{f}(z)\right| < \varepsilon$$

holds.

References

[1]. Stanković, B., Pilipović, S., Teorija distribucija, PMF, Novi Sad (1988).
[2]. Carmichael, R. D. and Mitrović, D., Distributions and analytic functions, John Wiley and Sons, Inc., New York (1988).
[3]. Schwartz, L., Theorie des distributions, Tome I, Tome II, Herman and Cie, Paris (1950, 1951).
[4]. Garnett, J., Bounded analytic function, Acad. Press, New York (1981).
[5]. J. Korevaar., Fourier Analysis and related topics, Amsterdam, Spring, 2011.
[6]. M. Avdispahic., On the determination of the jump of a function by its Fourier series, Acta Math. Hungar., 48(1986).