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Conference Paper

Introduction to Stochastic Differential Equations

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Abstract

Stochastic differential equations provide a link between probability theory and ordinary and partial differential equations. The stochastic modeler benefits from centuries of development of the physical sciences, and many classic results of mathematics can be given new intuitive interpretations.

There are several reasons why one should learn more about SDE: They have a wide range of applications outside mathematics, there are many fruitful connections to other mathematical disciplines and the subject has a rapidly developing life of its own as a fascinating research field with many interesting unanswered questions. The paper is meant to be an appetizer and is the answer of the issues:

- In what situations does the SDE arise?
- What are its essential features?
- What are the applications and the connections to other fields?

First we have constructed Wiener process, or Brownian motion. The study of Brownian motion can be intense, but the main ideas are a simple definition and the application of Itô Lemma. Then we will deal with the stochastic calculus and then we will show that SDEs can be solved. In the end we will see the modeling problems: How does SDEs model the physical situation and white noise process which is the generalized mean-square derivative of the Wiener process or Brownian motion.
Keywords: SDE, Brownian motion, Itô's formula, white noise

1. Introduction

Stochastic differential equations (SDEs) provide a link between probability theory and the much older and more developed fields of ordinary and partial differential equations. Wonderful consequences flow in both directions. The stochastic modeler benefits from centuries of development of the physical sciences, and many classic results of mathematics can be given new intuitive interpretations. The purpose of this paper is to provide an introduction to SDEs from applied point of view.

There are several reasons why one should learn more about SDE: They have a wide range of applications outside mathematics, there are many fruitful connections to other mathematical disciplines and the subject has a rapidly developing life of its own as a fascinating research field with many interesting unanswered questions.

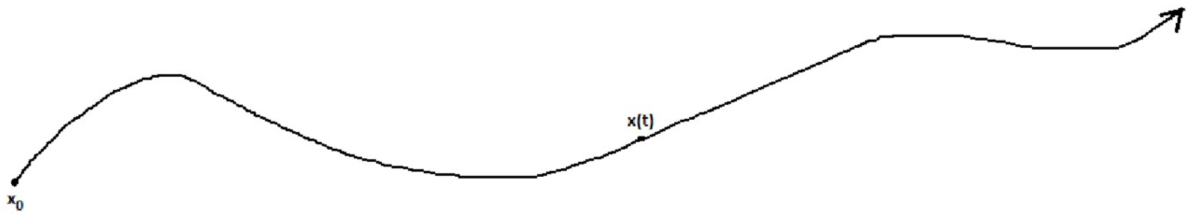
In this paper we would not be so interested in the proof of the most general case, but rather in an easier proof of a special case, which may give just as much of the basic idea in the argument. And we would be willing to believe some basic results without proof in order to have time for some more basic applications. This article reflects this point of view. Such an approach enables us to reach the highlights of the theory quicker and easier. The material has been selected with a view of being enough to bring out the main challenges of the topic but at the same time introductory enough to require only a modest mathematical background. So, the article is meant to be an appetizer. If it succeeds in awaking further interest, the reader will have a selection of excellent literature available for the study of the whole story.

First we have defined the Stochastic differential equations and their solutions:

Fix a point $x_0 \in R^n$ and consider then the ordinary differential equation:

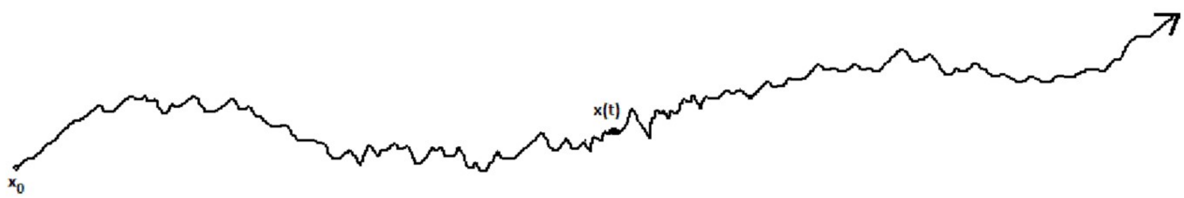
(ODE) $$\begin{cases} \frac{d}{d(t)}x(t) = b\big(x(t)\big) \quad (t > 0) \\ x(0) = x_0 \end{cases},$$

where b : $R^n \rightarrow R^n$ is a given, smooth vector field and the solution is the trajectory x(t) : [0,∞) → R^n.



Trajectory of the differential equation

In many applications, however, the experimentally measured trajectories of systems modeled by (ODE) do not in fact behave as predicted:



Sample path of the stochastic differential equation

Hence we can modify (ODE) to include the possibility of random effects disturbing the system:

(1) $$\begin{cases} \frac{dX(t)}{dt} = \mathrm{b}(\mathrm{X}(t)) + \mathrm{B}(\mathrm{X}(t))\xi(t) \qquad (t > 0) \\ \mathrm{X}(0) = x_0 \end{cases}$$

where **B** : $\mathbb{R}^n \rightarrow \mathbb{M}^{mxn}$ (= space of *n* × *m* matrices) and

ξ(t) := *m*-dimensional "white noise".

We will look below that the equation (1) has a solution and we will define the "white noise" *ξ*(**·**) too, but some mathematical problems about the SDE solution such as: the existence and uniqueness of solution, asymptotic behavior, dependence upon x_0 and other properties of solution, are not treated in this paper, because the purpose is to stop in the case when *m* = *n*, x_0 = 0, **b** ≡ 0, and **B** ≡ *I*. The solution of (1) in this setting turns out to be the *n*-dimensional *Wiener process*, or *Brownian motion*, denoted W(**·**). Thus we may symbolically write

$\dot{W}(\cdot) = \xi(\cdot)$,

thereby asserting that *"white noise" is the time derivative of the Wiener process*.

If we multiply (1) by "*dt*":

(*SDE*) $$\begin{cases} dX(t) = \mathrm{b}(\mathrm{X}(t))\mathrm{dt} + \mathrm{B}\big(\mathrm{X}(t)\big)dW(t) \\ \mathrm{X}(0) = x_0 \end{cases}$$

This expression, properly interpreted, is a *stochastic differential equation*. We say that **X**(**·**) *solves* (SDE) provided

(2) $$\mathbf{X}(t) = x_0 + \int_0^t \mathrm{b}\big(\mathrm{X}(s)\big)ds + \int_0^t \mathrm{B}\big(\mathrm{X}(s)\big)dW \qquad \text{for all times } t > 0 .$$

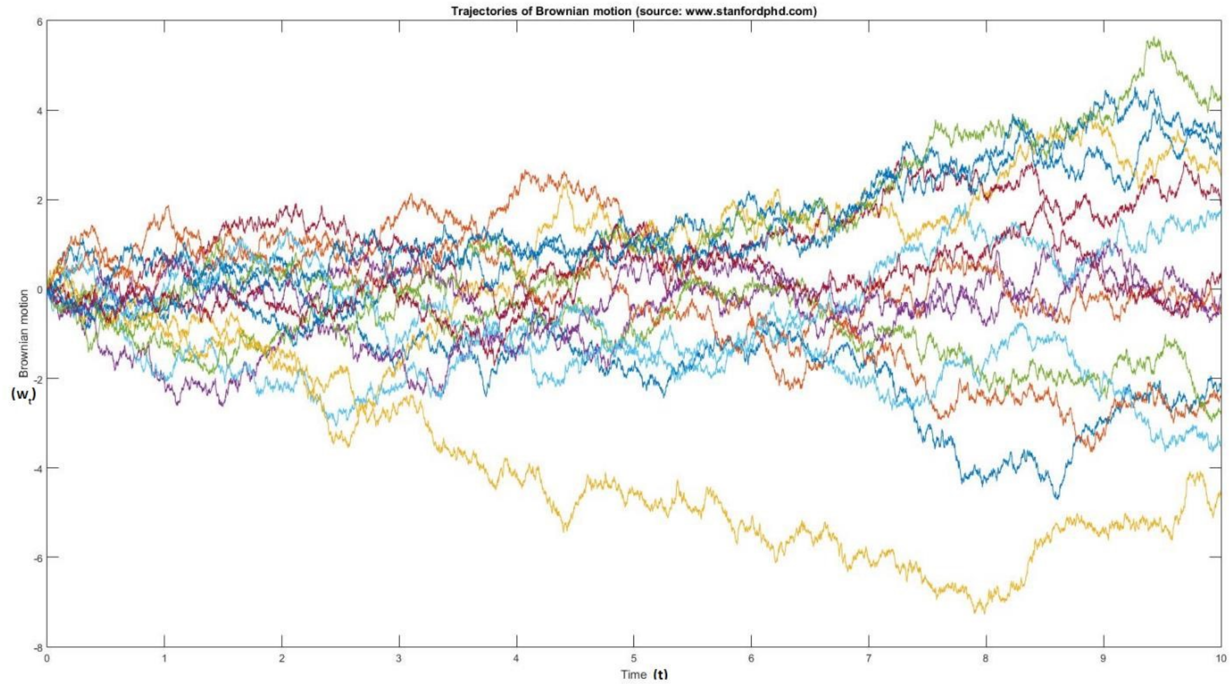


Fig. 1: Trajectories of Brownian Motions

So, we will define the Brownian motion and stochastic calculus (Itô Lemma) and then we will see the modeling problems:

• Does (SDE) truly model the physical situation?

• Is the term $\xi(\cdot)$ in (1) "really" white noise, or is it rather some ensemble of smooth, but highly oscillatory functions?

As we will see later these questions are subtle, and different answers can yield completely different solutions of (SDE).

2.1. Brownian Motion & Stochastic Calculus

An important example of SDEs is Brownian Motion. The study of Brownian motion can be intense, but the main ideas are a simple definition and the application of Itô's lemma.

Definition and Properties of Brownian Motion

On the time interval [0, T], Brownian motion is a continuous stochastic process $(W_t)_{t \le T}$ such that

1. $W_0 = 0$,
2. Independent Increments: for $0 \le s' < t' \le s < t \le T$, $W_t - W_s$ is independent of $W_{t'} - W_{s'}$,
3. Conditionally Gaussian: $W_t - W_s \sim N(0, t-s)$, i.e. is normal with mean zero and variance t − s.

There is a vast study of Brownian motion. We will instead use Brownian motion rather simply; the only other fact that is somewhat important is that Brownian motion is nowhere differentiable, that is

$\mathbb{P}\left(\frac{d}{dt}W_t \text{ is undefined for almost-everywhere } t \in [0, T]\right) = 1$

although we can write $\dot{W}_t$ to denote a white noise process. It should also be pointed out that W_t is a martingale,

$\mathbb{E}\{W_t \mid (W_\tau)_{\tau \le s}\} = W_s \qquad \forall s \le t.$

Simulation. There are any number of ways to simulate Brownian motion, but to understand why Brownian motion is consider a 'random walk', consider the process

$$W_{t_n}^N = W_{t_n}^N + \begin{cases} \sqrt{T/N} & \text{with probability } 1/2 \\ -\sqrt{T/N} & \text{with probability } 1/2 \end{cases}$$

with $W_{t_0} = 0$ and $t_n = n\frac{T}{N}$ for n = 0, 1, 2 . . . , N. Obviously $W_{t_n}^N$ has independent increments, and conditionally has the same mean and variance as Brownian motion. However it's not conditional Gaussian. However, as N → ∞ the probability law of $W_{t_n}^N$ converges to the probability law of Brownian motion, so this simple random walk is actually a good way to simulate Brownian motion if you take N large. However, one usually has a random number generator that can produce $W_{t_n}^N$ that also has conditionally Gaussian increments, and so it probably better to simulate Brownian motion this way. A sample of independent Brownian Motions simulations is given in Figure 1.

2.2 The Itô Integral

First we need to understand the definition of an Itô (stochastic) integral, and then we can understand what it means for a stochastic process to have a differential. The construction of the Itô integral begins with a backward Riemann sum. For some function f : [0, T] → ℝ (possibly random), non-anticipative of W, the Itô integral is defined as

$$\int_0^T f(t)dW_t = \lim_{N\to 0} \sum_{n=0}^{N-1} f(t_n)\left(W_{t_{n+1}} - W_{t_n}\right) \qquad (1)$$

where $t_n = n\frac{T}{N}$, with the limit holding in the strong sense.

Another important property of the stochastic integral is the Itô's Isometry,

Proposition (Itô's Isometry). For any functions f, g (possibly random), nonanticipative of W, with $\mathbb{E}\int_0^T f^2(t)dt < \infty$ and $\mathbb{E}\int_0^T g^2(t)dt < \infty, then$

$$\mathbb{E}\left(\int_0^T f(t)dW_t\right)\left(\int_0^T g(t)dW_t\right) = \mathbb{E}\int_0^T f(t)g(t)dt$$

Some facts about the Itô's *integral:*

• One can look at Equation (1) and think about the stochastic integral as a sum of independent normal random variables with mean zero and variance *T /N*,

$$\sum_{n=0}^{N-1} f(t_n)\left(W_{t_{n+1}} - W_{t_n}\right) \sim \mathcal{N}\left(0, \frac{T}{N}\sum f^2(t_n)\right).$$

Therefore, one might suspect that $\int_0^T f(t)dW_t$ is normal distributed.

• In fact, the Itô integral is normally distributed when f is a non-stochastic function, and its variance is given by the Itô isometry:

$$\int_0^T f(t)dW_t \sim \mathcal{N}\left(0, \int_0^T f^2(t)dt\right).$$

• The Itô integral is also defined for functions of another random variable. For instance, $f: \mathbb{R} \to \mathbb{R}$ and another random variable X_t, the Itô integral is

$$\int_0^T f(X_t)dW_t = \lim_{N\to\infty} \sum_{n=0}^{N-1} f\left(X_{t_n}\right)\left(W_{t_{n+1}} - W_{t_n}\right).$$

The Itô isometry for this integral is

$$\mathbb{E}\left(\int_0^T f(X_t)dW_t\right)^2 = \int_0^T \mathbb{E}\, f^2(X_t)\, dt,$$

provided that *X* is non-anticipative of *W*.

Given a stochastic differential equation, the Itô lemma tells us the differential of any function on that process. Itô's lemma can be thought of the stochastic analogue to differentiation, and is a fundamental tool in stochastic differential equations:

Itô Lemma: Consider the process X_t with SDE $dX_t = a(X_t)dt + b(X_t)dW_t$. For a function $f(x,t)$ with at least one derivative in *t* and at least two derivatives in *x*, we have

$$df(t,X_t) = \left(\frac{\partial}{\partial t} + a(X_t)\frac{\partial}{\partial x} + \frac{b^2(X_t)}{2}\frac{\partial^2}{\partial x^2}\right) f(t,X_t)dt + b(X_t)\frac{\partial}{\partial x} f(t,X_t)\, dW_t.$$

2.3 Differential equations with driving White Noise

Many time-varying phenomena in various fields in science and engineering can be modeled as differential equations of the form

$$\frac{dx}{dt} = f(x,t) = L(x,t)\, w(t) \qquad (2.3.1)$$

where $w(t)$ is some vector of forcing functions. We can think a stochastic differential equation (SDE) as an equation of the above form where the forcing function is a stochastic process. One motivation for studying such equations is that various physical phenomena can be modeled as random processes (e.g., thermal motion) and when such a phenomenon enters a physical system, we get a model of the above SDE form. Another motivation is that in Bayesian statistical modeling unknown forces are naturally modeled as random forces which again leads to SDE type of models. Because the forcing function is random, the solution to the stochastic differential equation is a random process as well. With a different realization of the noise process we get a different solution. For this reason the particular solutions of the equations are not often of interest, but instead, we aim to determine the statistics of the solutions over all realizations. An example of SDE solution is given in Figure 2

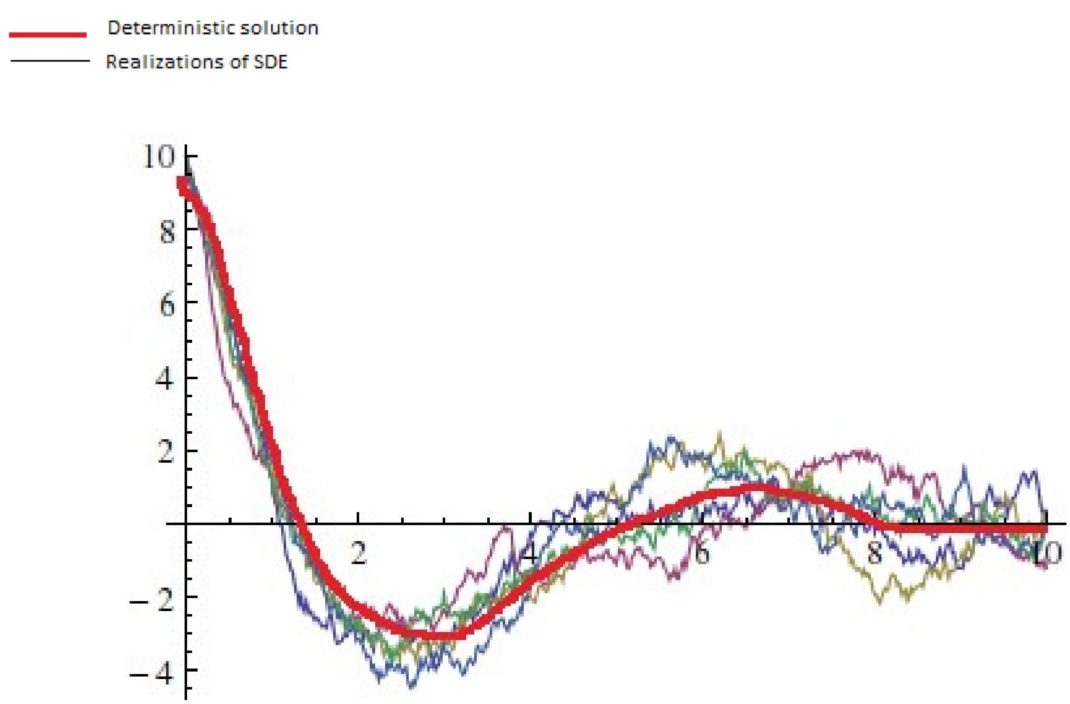


Fig. 2: The solution of the SDE is different for each realization of noise process. We can also compute the mean of the solutions, which in the case of linear SDE corresponds to the deterministic solution with zero noise.

In the context of SDEs, the term $f(x,t)$ in Equation (2.3.1) is called the drift function which determines the nominal dynamics of the system, and $L(x,t)$ is the dispersion matrix which determines how the noise $w(t)$ enters the system.

This indeed is the most general form of SDE that we discuss in the document. Although it would be tempting to generalize these equations to $dx/dt = f(x, w, t)$ it is not possible in the present theory. The unknown function usually modeled as Gaussian and "white" in the sense that $w(t)$ and $w(\tau)$ are uncorrelated (and independent) for all t ≠ s. The term white arises from the property that the power spectrum (or actually, the spectral density) of white noise is constant (flat) over all frequencies. White light is another phenomenon which has this same property and hence the name. In mathematical sense white noise process can be defined as follows:

Definition of White Noise:

White noise process $w(t) \in \mathbb{R}^s$ is a random function with the following properties:

1. $w(t_1)$ and $w(t_2)$ are independent if $t_1 \neq t_2$
2. $t \mapsto w(t)$ is a Gaussian process with zero mean and Dirac-delta-correlation:

$$m_w(t) = E[w(t)] = 0$$
$$C_w(t, s) = E[w(t)\, w^T(s)] = \delta(t - s)Q,$$

where Q is the spectral density of the process.

From the above properties we can also deduce the following somewhat peculiar properties of white noise:

- The sample path $t \mapsto w(t)$ is discontinuous almost everywhere.
- White noise is unbounded and it takes arbitrarily large positive and negative values at any finite interval.

An example of a scalar white noise process realization is shown in Figure 3.

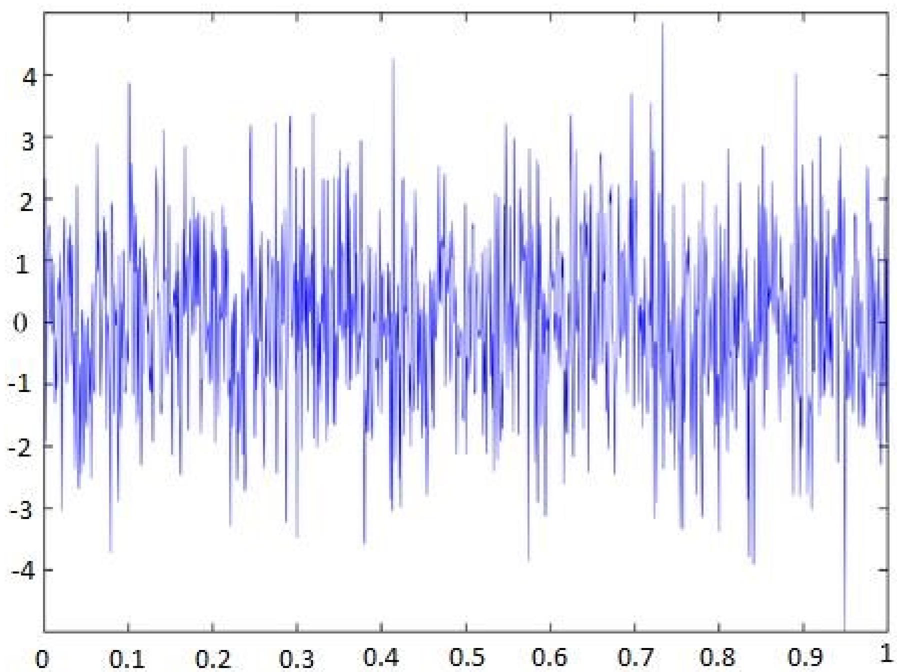


Figure 3. White noise

It is also possible to use non-Gaussian driving functions in SDEs, but here we will always assume that the driving function is Gaussian.

3. Conclusions

In this paper we pointed out that SDEs have a wide range of applications outside mathematics. For example continuous time financial models will often use Brownian motion to model the trajectory of asset prices. One can typically open the finance section of a newspaper and see a time series plot of an asset's price history, and it might be possible that the daily movements of the asset resemble a random walk or a path taken by a Brownian motion. Such a connection between asset prices and Brownian motion was central to the formulas of [Black and Scholes, 1973] and [Merton, 1973], and have since led to a variety of models for pricing and hedging. The study of Brownian motion and the applications of SDE in the finance is a theme we would like to treat in the future; using MATLAB to show applications of SDE in financial modeling.

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