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Conference Paper

A note on the cluster sets of interpolating Blaschke products

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Abstract

In this paper we are going to represent Blaschkeproducts, where Blaschke product with zero sequence (a_n) in the open unit disk D is a function of the form $B(z)=e^{i\theta}z^N\prod_{n=1}^{\infty}\frac{\bar{a}_n}{|a_n|}\frac{a_n-z}{1-\bar{a}_n z}$, where (a_n) satisfy $\sum_n(1-|a_n|)<\infty$.

In this research we are going to analyze specifically interpolating Blaschke product, for which the zero sequence is an interpolating sequence and notes about radial cluster set. In the second part we are going to show the existence of interpolating Blaschke products that have intervals a radial cluster set, but the main result of this research is that there does not exist an interpolating Blaschke product having [0*, y*] or [*x,* 0] as a radial cluster set. On the other hand, there does not exist an interpolating Blaschke product with [0, 1] as radial clusterset.

Keywords: Blachke products, radial cluster set, interpolating Blaschke products, interpolating sequence.

1. Introduction

1.1. The cluster set

The notion of a cluster set was first formulated explicitly by Painleve in 1895. Painleve introduced the cluster set, which he called domained'indetermination, as a descriptive notion to characterize in an intuitive way the behaviour of an analytic function in the neighbourhood of a singularity in terms of the properties of the set of all its limits at the singularity, and to classifythe singularities of a function in terms of these cluster sets.

The unite disc is denoted by $D=\{z:|z|<1\}$, the boundary of D is denoted by $C=\{z:|z|=1\}$.

Let the function $\omega=f(z)$ be defined in a complex domain Ω which it maps into the Rimman's ω -sphere S. Ω is connected, the boundary of Ω is denoted by $\partial\Omega$, while $\overline{\Omega}=\Omega\cup\partial\Omega$ is the closure of Ω.

Definition1.1.1.*Given any point $z_0\in\overline{\Omega}$, we may define the* ***cluster set*** *$C_\Omega(f,z_0)$ of $f(z)$ in either of the following two equivalent ways:*

(i) $C_\Omega(f,z_0)$ is the set of points α of the ω -sphere S such that there exists a sequence $(z_n)\subseteq\Omega-\{z_0\}$ such that $\lim_{n\to\infty}z_n=z_0$ and $\lim_{n\to\infty}f(z_n)=\alpha$.

(ii) $C_\Omega(f,z_0)=\bigcap_{r>0}\overline{\Omega}_r$, where $\Omega_r=f(\delta_r\cap(\Omega\setminus\{z_0\}))$ and δ_r is the disc $|z-z_0|<r$.

It follows immediately from the definition in either form that $C_\Omega(f,z_0)$ is a non-empty closed set. A cluster set which contains only a single point will be called *degenerate*, while if it contains more than one point, it will be called *total* or *subtotal*.

The theory of cluster sets has been built up round three situations which are geometrically the simplest: namely

(a) the case where z_0 is anisolated frontier point orisan interior point of Ω ;

(b) the case where Ω is the unit disc $D=\{z:|z|<1\}$ and z_0 a frontier point;

(c) the case where $\partial\Omega$ is totally disconnected and $z_0 \in \partial\Omega$.

Definition1.1.2. *Let ζ be any infinite subset of Ω on the real line or in the z-plane, and let $z_0 \in \zeta'$ where $z_0 \in \zeta'$ is the derived set of ζ. Then define the cluster set $C_\zeta(f,z_0)$ of $f(z)$ on ζ by :*

$$C_\zeta(f,z_0)=\bigcap_{r>0}\overline{\Omega_r(\zeta)}\subset C_\Omega(f,z_0)$$

where $\Omega_r(\zeta)=f\left(\delta_r\cap\left(\zeta\setminus\{z_0\}\right)\right)$ *is the image of* $\delta_r\cap\left(\zeta\setminus\{z_0\}\right)$ *on* S.

$C_\Omega(f,z_0)$ *is the **complete cluster set** of* $f(z)$ *at* z_0

$C_\zeta(f,z_0)$ *is a **partial cluster set** of* $f(z)$ *at* z_0 .

We are going to show some classical theorems from the theory of cluster sets.

Theorem 1.1.1.(Theorem of Weierstrass and Painleve)

Let z_0 be an isolated point of the set E, $E\subseteq\Omega$, if $f(z)$ is meromorphic in $\Omega\setminus E$, then $C(f,z_0)$ is either total or degenerate.

In the usual terminology, we shall say that a set E in the plane is of linear measure zero if E can be covered by a sequence of circlesthe sum ofwhose diameters can be made arbitrarily small.

Theorem 1.1.2. (Fatou)*If $f(z)=\sum_{n=0}^{\infty}a_n z^n$ is analytic and bounded in unite disc D, i.e.$|f(z)|<M<\infty$,in D, then the radial limits $f(e^{i\theta})=\lim_{r\to 1}f(re^{i\theta})$ exist for all points $e^{i\theta}$ on C, except possibly for a set of linearmeasure zero.*

Theorem 1.1.3.(F.and M. Riesz)

If $f(z)$ is analytic and bounded, $|f(z)|<M<\infty$, in D ,and if the set E of points $e^{i\theta}$ for which $f(e^{i\theta})=\lim_{r\to 1}f(re^{i\theta})=0$ has positive measure on C ,then $f(z)$ is identically zero,i.e. $f(z)\equiv 0$ in D.

Suppose $F\subset H(\Omega)$ for some region (i.e. open connected) Ω. (H(Ω) means the set of all holomorphic function in Ω)

Definition1.1.3.*We call **F normal family** if every sequence of members of F contains a subsequence which converges uniformly on compact subsets of Ω.*

Let X be a topological space. We denote by C(X) the set of complex valued continuous functions on X.

Theorem 1.1.4. (Hurwitz)

Let $D \subseteq C$ be a region and $\{f_n\}$ $n\in N$ a sequence of injective functions $f_n\in H(D)$ (holomorphic functions) converging uniformly in every compact subset of D to f. Then, either f is constant or f is injective.

1.2.Blaschke products and interpolating Blaschke products

We consider in this part an important class of functions which are analytic and bounded in unite disc D .

Definition 1.2.1.*ABlaschke productwith zero sequence (an) in the open unit disk D is a function of the form* $B(z)=e^{i\theta}z^N\prod_{n=1}^{\infty}\frac{\bar{a}_n}{|a_n|}\frac{a_n-z}{1-\bar{a}_n z}$ *, where* (a_n) *satisfy* $\sum_n(1-|a_n|)<\infty$.

The function B is called *normalized*, if $B(0)>0$. Wedenote the zero set of $B$by$Z(B)$.

The class of functions known as Blaschke products is distinguished by many interesting features, the first of these properties, due toF. Riesz, is that the radial limit values, which exist almost everywhere by Fatou's theorem, are of modulus 1 almost everywhere on C .Thus we have

Theorem 1.2.1.*A Blaschke product possesses radial limits of modulus 1 for almost all* $e^{i\theta}$ *on* C .

Theorem 1. 2.2. *A necessary and sufficient condition that a function* $f(z)$, *analytic and bounded in* D , $|f(z)|<1$ *, be a Blaschke product is that:*

$$\lim_{r\to 1}\int_0^{2\pi}\log\left|f\left(re^{i\theta}\right)\right|d\theta=0 .$$

A sequence (a_n) in D is called an *interpolating sequence* if for every bounded sequence of complex numbers (w_n) ,there exists a bounded analytic function f in D such that $f(a_n)=w_n$ for every $n\in\mathbb{N}$.

In 1958, Carleson [5] proved that a sequence of points (a_n) in D is an interpolating sequence if and only if

$$\inf_{k\in\mathbb{N}}\prod_{j:j\neq k}\left|\frac{a_j-a_k}{1-\bar{a}_j a_k}\right|\geq\delta>0 .$$

Definition 1.2.2.A Blaschke product for which the zero sequence is an interpolating sequence is called an *interpolating Blaschke product* with *uniform separation constant* $\delta(B)$ defined by

$$\delta(B):=\inf_{k\in\mathbb{N}}\prod_{j:j\neq k}\left|\frac{a_j-a_k}{1-\bar{a}_j a_k}\right|$$

We note that $\delta(B)=\inf_n\left(1-|a_n|^2\right)|B'(a_n)|$.

By convention, a finite Blaschke product with simple zeros will be considered an interpolating Blaschke product with associated separation constant as defined above.

Since it is easy for a sequence to satisfy the Blaschke condition $\sum_n(1-|a_n|)<\infty$, and because it appears to be difficult to satisfy Carleson's condition, it may seem that there are far more Blaschke products than interpolating Blaschke products.

A Blaschke product with zero sequence (a_n) is called *thin*, if $\lim_{n\to\infty}\left(1-|a_n|^2\right)|B'(a_n)|=1$.

It is easy to see that any thin Blaschke product B can be written as $B=pb$ with $Z(p)\subseteq Z(b)$ where p is a finite Blaschke product and b is an interpolating Blaschke product.

T. Wolff [6]showed that *thin sequences* are the universal interpolating sequences for the algebra of bounded analytic functions. Since these sequences are so special, it is nice to be able to replace results for Blaschke productswith results for thin Blaschke products. Thus, if we can obtain a thin Blaschke product that has a specified radial cluster set.

As usual, we let $\rho(z,w)=\left|\frac{z-w}{1-\bar{z}w}\right|$ be the *pseudohyperbolic distance* between two points, *z* and *w*,in D. The pseudohyperbolic disk centered at the point *a*and of radius*r* is denoted by $D_\rho(a,r)$ and is defined by $D_\rho(a,r)=\{z\in D:\rho(a,z)\le r\}$.

We recall here a useful result due to K. Hoffman about the constants associated with interpolating Blaschke products.

Hoffman's Lemma 1. 2.1. *Let* δ, η *and* ε *be real numbers satisfying* $0<\delta<1$, $0<\eta<\left(1-\sqrt{1-\delta^2}\right)/\delta$ *,(* $0<\eta<\rho(\delta,\eta)$ *) and* $0<\varepsilon<\eta\frac{\delta-\eta}{1-\delta\eta}$.

If B is any interpolating Blaschke product with zeros $\{z_n\}$ *such that*

$$\delta(B)=\inf_{n\in N}\left(1-|z_n|^2\right)|B'(z_n)|\ge\delta$$

then

1. the pseudo hyperbolic disks $D_\rho(a,\eta)$ *for a* $a\in Z(B)$ *are pairwise disjoint.*

2. The following inclusions hold:

$$\{z\in D:|B(z)|<\varepsilon\}\subseteq\{z\in D:\rho(z,Z(B))<\eta\}\subseteq\{z\in D:|B(z)|<\eta\}$$

3.The set $\{z\in D:|B(z)|<\varepsilon\}$ *can be written as a union of pairwise disjoint domains* V_n *with* $z_n\in V_n$ *and* $V_n\subseteq D_\rho(z_n,\eta)$ *. Moreover, the Blaschke product B is a bijection of* V_n *onto* $\{w\in D:|w|<\varepsilon\}$

4. For every a *with* $|a|<\varepsilon$ *the Frostman shift* $B_a=\frac{B-a}{1-\bar{a}B}$ *of B is an interpolating Blaschke product having a unique zero in each* V_n *and satisfying*

$$\delta(B_a)\ge\left[\delta-2\eta/\left(1+\eta^2\right)\right]/\left[\left(1-\delta2\eta\right)/\left(1+\eta^2\right)\right]$$

5. If B *is a thin Blaschke product, then for every* $a\in D$ *, the function* $B_a=\frac{B-a}{1-\bar{a}B}$ *is also a thin Blaschke product.*

We collect some information about the constants δ, η and ε here. We note that $1-\sqrt{1-\delta^2}/\delta$ is a monotonically increasing function of $\delta\in[0,1]$ that $\varepsilon<\eta<\delta$ and that $0<1-\sqrt{1-\delta^2}/\delta<\delta$ Moreover, $\eta<2\eta/\left(1+\eta^2\right)<\delta$ is equivalent to $0<\eta<\rho(\delta,\eta)$.

Recall that the radial cluster set of a function $f\in H^\infty$ at the point $e^{i\theta}$ is the set of all values w , for which there exists a sequence (r_n) of points in $[0,1]$ satisfying $f\left(r_ne^{i\theta}\right)\to w$.

The radial cluster set of *f* at the point $e^{i\theta}$ will be denoted by $C_r\left(f,e^{i\theta}\right)$.

Radial cluster sets of innerfunctions are compact, connected subsets of the closed unit disk.

Recall that the *N*-th *tail* of the Blaschke product $B=\prod_{n=1}^{\infty}\frac{\bar{a}_n}{|a_n|}\frac{a_n-z}{1-\bar{a}_nz}$ is given by $\prod_{n=N}^{\infty}\frac{\bar{a}_n}{|a_n|}\frac{a_n-z}{1-\bar{a}_nz}$.

Proposition 1.2.1.*Let B be a thin Blaschke product. Then for every* $e^{i\theta} \in \partial D$ *we have*

$$\limsup_{r\to 1}\left|B\left(re^{i\theta}\right)\right| = 1$$

Proposition 1.2.2. *Let B be a thin Blaschke product with positive zeros. Then the radial cluster set of B at the point z* = 1 *is the interval* [−1, 1].

2. Main Results

Thinking about the result above naturally leads to the question of which intervals can be the radial cluster of an interpolating Blaschke product. For example, if we consider *B* as in the previous theorem, we obtain a Blaschke product with radial cluster set [0, 1]; namely, $C_r\left(B^2,1\right)=\left[0,1\right]$. The question is if the interpolating Blaschkeproduct exist with radial cluster set equal to [0, 1]? The answer, presented in our next theorem, is "no".

Theorem 2.1.*There does not exist an interpolating Blaschke product with* [0, 1] *as radial clusterset.*

Proof. Suppose there exists an interpolating Blaschke product b for which $C_r\left(b,1\right)=\left[0,1\right]$.
Let δ be the uniform separation constant of*b*and let $Z(b)=\{z_n : n \in \mathrm{N}\}$be the zero set of *b*.

ByHoffman's Lemma for every η satisfying $0<\eta<\dfrac{1-\sqrt{1-\delta^2}}{\delta}$, there exists ε such that $0<\varepsilon<\eta$ and $\left|b\left(z\right)\right|\geq\varepsilon$ whenever $\rho\left(z,Z\left(b\right)\right)\geq\eta$. Moreover, $\left\{\left|b\right|<\varepsilon\right\}$ is a union of pairwise disjoint domains $V_n \subseteq D_\rho\left(z_n,\eta\right)$ such that *b* is a bijective mapping of V_n onto $\left\{\left|w\right|<\varepsilon\right\}$ for every *n*.
Since $D_\rho\left(z_n,\varepsilon\right)\subseteq V_n$ we may assume, by passing to smaller $\eta's$, that *b* actually is injective on $D_\rho\left(z_n,\dfrac{2\eta}{1+\eta^2}\right)$

Hence the preimage, with respect to *b*, of the circle $\left|w\right|=\varepsilon$ is a disjoint union of Jordan arcs J_n, surrounding the disks $D_\rho\left(z_n,\varepsilon\right)$.
Since $0\in C_r\left(b,1\right)$,there is a sequence $\left(r_n\right)$ with $r_n\in\left[0,1\right]$ and $b\left(r_n\right)\to 0$.
We may, of course, assume that $\left|b\left(r_n\right)\right|<\varepsilon$.| Hence, there exist $z_{k(n)}\in Z\left(b\right)$ such that $\rho\left(r_n,z_{k(n)}\right)\to 0$ as $n\to\infty$.

Thus for *n* sufficiently large, say $n\geq n_0$, the boundary of the disk $D_\rho\left(z_{k(n)},\varepsilon\right)$ cuts the real axis twice. Therefore, for $n\geq n_0$, the Jordan arcs J_n meet the real line at least twice.
Note that $r_n\in D_\rho\left(z_{k(n)},\varepsilon\right)\subseteq V_{k(n)}$. Hence there exists one intersection point, called x_n ,of this Jordan curve J_n with $\left[0,1\right]$, that lies to the left of r_n and there exists an intersection point y_n , that lies to the right of r_n .
We will show that there exists $n_1>n_0$ such that for $n\geq n_1>n_0$ we have

$$\varepsilon/2 < \rho(x_n, y_n) < \frac{2\eta}{1+\eta^2}$$

Indeed, we know that $\rho(x_n, z_{k(n)}) \geq \rho(b(x_n), b(z_{k(n)})) = |b(x_n)| = \varepsilon$ and so $\rho(x_n, r_n) + \rho(r_n, z_{k(n)}) \geq \rho(x_n, z_{k(n)}) \geq \varepsilon$. Since $\rho(x_n, z_{k(n)}) \to 0$,there exists $n_1 > n_0$ such that $\rho(x_n, r_n) \geq \varepsilon/2$ for all $n \geq n_1$.

Now we use the fact that $\rho(x_n, y_n) \geq \rho(x_n, r_n)$to obtain the lower estimate. The upper estimate is a consequence of the triangle inequality $\rho(u,v) \leq \dfrac{\rho(u,w)+\rho(w,v)}{1+\rho(u,w)\rho(w,v)}$ and the fact that $\rho(x_n, z_{k(n)}) < \eta$ and $\rho(y_n, z_{k(n)}) < \eta$.

Now choose ξ_n and τ_n in D with $L_{z_{k(n)}}(\xi_n) = x_n$ and $L_{z_{k(n)}}(\tau_n) = y_n$. By the invariance of the ρ -distance under conformal automorphisms of the unit disk, $\varepsilon/2 < \rho(\xi_n, \tau_n) < \dfrac{2\eta}{1+\eta^2}$.

Further $|\xi_n| = \rho\left(L^{-1}_{z_{k(n)}}(x_n), L^{-1}_{z_{k(n)}}(|z_{k(n)}|)\right) = \rho(x_n, |z_{k(n)}|) \leq \rho(x_n, z_{k(n)}) < \eta$.

Repeating this argument, we also get $|\tau_n| \leq \eta$. Let ξ and τ be cluster points in D of the sequences(ξ_n) and (τ_n), respectively. Then $\rho(\xi, \tau) \geq \varepsilon/2$. A normal families argument yields a subsequence of $\left(b \circ L_{z_{k(n)}}\right)_n$, denoted here by $b \circ L_n$,converging locally uniformly in D to a function $f \in H^\infty$. The functions $b \circ L_n$ are injective on $\left\{z : |z| < \dfrac{2\eta}{1+\eta^2}\right\}$,and therefore by Hurwitz'stheorem, their limit function *f* is either injective or constant. But *b*is interpolating and therefore $\left|(b \circ L_n)'(0)\right| \geq \delta$, for each *n*. Hence *f*is not constant, and therefore *f* is injective on $\left\{z : |z| < \dfrac{2\eta}{1+\eta^2}\right\}$.

We conclude that $f(\xi) \neq f(\tau)$, and therefore there exists $d > 0$ with $|f(\xi) - f(\tau)| > d > 0$.

Recall that x_n and y_n were chosen so that $|b(x_n)| = |b(y_n)| = \varepsilon$.Hence there exists a subsequence(n') of $k(n)$ such that $|b(x_{n'}) - b(y_{n'})| \geq d > 0$ and $|b(x_{n'})| = |b(y_{n'})| = \varepsilon$. Thus, there exist atleast two different points in $C_r(b,1)$ having modulus ε.

Therefore $C_r(b,1) \neq [0,1]$. □

Let $x \in [0,1]$, if we choose in the proof above $\varepsilon < x$,then we see that there does not exist an Interpolating Blaschke product with radial cluster set $[0, x]$.

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