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Some results on Greens relations in direct sum and product of rings.

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Abstract

Green's relations in a semigroup are successful tools for studying their properties. These relations are introduced and studied also in rings. It is wellknown that every ring can be represented as a sub-direct sum of subdirectly irreducible rings. Our aim in this paper is to find the connection of Green's relations on direct sum and direct product of rings

Keywords: *Green's relations, Rings, Semigroup, Direct sum of rings, Direct product of rings.*

related to Green's relations in its components.

In this paper by a ring we mean an associative ring, which does not neccessary have identity element. The relations $\mathcal{L}$ $\mathcal{R}$ and $\mathcal{D}$ in rings were first introduced and studied by Petro .

This relations are called Green's relations in rings because they mimic the relations $\mathcal{L},\mathcal{R}$ and $\mathcal{D}$ in semigroups,

1. Introduction

which were first introduced and studied by James Alexander Green (Green, 1951).
We give some notions and present some auxiliary results that will be used throught the paper. Some of the results and other basic notions may be found in [1], [5], [6]. First we give the definitions of Green's relations ,

Let A be a ring and let $a \in A$. The principal left (right) ideal $((a)_l,(a)_r)$ generated by a is
$+$ $+$, were I denotes the ring of all integers.

Definition 1. [2] *Green's relations* $\mathcal{L}$ *and* $\mathcal{R}$ *in ring* A *are defined by:*

$$a\mathcal{L}b \Leftrightarrow (a)_l = (b)_l,$$

$$a\mathcal{R}b \Leftrightarrow (a)_r = (b)_r.$$

$\mathcal{L}$ $\mathcal{R}$ on rings.

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It is evident that $\mathcal{L}$ and $\mathcal{R}$ are equivalence relations. Let $\mathcal{L}_a(\mathcal{R}_a)$ be the equivalence class of a

For sake of simplicity we use the following notations:

$$kx + ux = (k,u)x, \quad k \in I, \quad x,u \in A,$$

$mod\,\mathcal{L}(mod\,\mathcal{R})$ containing the element $a \in A$.

$$m'x + yv' = y(m', v'), \quad m' \in I, \quad y, v' \in A.$$

Lemma 1.[2] *Let* A *be a ring and* $a, b, s \in A$. *Let* $k \in I$, *then the following implications hold.*

$$a\mathcal{L}b \Rightarrow a(k,s)\mathcal{L}b(k,s),$$
$$a\mathcal{R}b \Rightarrow (k,s)a\mathcal{R}(k,s)b.$$

Leme 2.[2] *The Green's relations* $\mathcal{L}$ *and* $\mathcal{R}$ *in a ring* A *commute.*

The join $R \vee L$ is also of great importance and we denote it by $\mathcal{D}$. The above Lemma shows that $\mathcal{L} \circ \mathcal{R}$ is an equivalence relations and also $\mathcal{D} = \mathcal{L} \vee \mathcal{R} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$

The intersection $H = \mathcal{L} \cap \mathcal{R}$ of the equivalence relations $\mathcal{L}$ and $\mathcal{R}$ on the ring A is an equivalence relation on A. We denote by $\mathcal{H}_a$ the equivalence class mod $\mathcal{H}$ containing a. Also for principal ideals ,we can define in a similar way as $\mathcal{R}, \mathcal{L}$ another Green's relation which we denote it by $\mathcal{J}$. Throughout this paper ,Green's relations in the multiplicative semigroup $(A, .)$ of the ring $(A, +, .)$ are denoted by $\mathcal{R}(.), \mathcal{L}(.), \mathcal{H}(.), \mathcal{D}(.), \mathcal{J}(.)$, in order to distinguish them from Green's relations $\mathcal{R}, \mathcal{L}, \mathcal{H}, \mathcal{D}, \mathcal{J}$ in the ring $(A, +, .)$.

For the same purpose the equivalence classes of an element a respect to Green's relations in the multiplicative semigroup of the ring $(A, +, .)$ are denoted $\mathcal{R}_a(.), \mathcal{L}_a(.), \mathcal{H}_a(.), \mathcal{D}_a(.), \mathcal{J}_a(.)$.

The direct product construction in modules is very straightforward and probably familiar to the reader, the same hold in ring theory for rings considering it as a module.

Definition 2.[1] *The direct product of a set* $\{A_i : i \in I\}$ *of rings denoted* $\prod_{i \in I} A_i$ *is the "cartesian product " which is a ring endowed with componentwise operations* $(a_i) + (a_i') = (a_i + a_i')$ *and* $(a_i)(a_i') = (a_i a_i')$ *where* $a_i, a_i' \in A_i$ *for all* i.

Definition 3.[1] *The direct sum* $\sum A_i$ *of a set* $A_i : i \in I$ *of rings is the set* $\{(a_i)_{i \in I} \in \prod_{i \in I} A_i :$ *almost all* a_i *are* $0\}$.

Thus the direct sum and direct product of $\{A_i : i \in I\}$ are the same if I is finite.

Definition 4.[4] *A ring* A *is left (right) s-unital if for every element* a *, is left (right) s-unital, i.e.*

$$\forall a \in A \Rightarrow a \in Aa (a \in aA) \quad .$$

Main results

Proposition 1. *Let* A_i *, for* $i \in I$ *be rings such that* $(a_i)_{i \in I}, (b_i)_{i \in I} \in \prod_{i \in I} A_i$ *then the following implications hold*

a) $(a_i)_{i \in I} \mathcal{J} (b_i)_{i \in I} \Rightarrow a_i \mathcal{J} b_i, \ \forall i \in I$,

b) $(a_i)_{i \in I} \mathcal{R} (b_i)_{i \in I} \Rightarrow a_i \mathcal{R} b_i, \forall i \in I$,

c) $(a_i)_{i \in I} \mathcal{L} (b_i)_{i \in I} \Rightarrow a_i \mathcal{L} b_i \forall i \in I$,

d) $(a_i)_{i \in I} \mathcal{D} (b_i)_{i \in I} \Rightarrow a_i \mathcal{D} b_i \forall i \in I$,

e) $(a_i)_{i\in I}\mathcal{H}(b_i)_{i\in I} \Rightarrow a_i\mathcal{H}b_i \forall i\in I$,

Proof. a) If $(a_i)\mathcal{J}(b_i) \Rightarrow ((a_i)_{i\in I}) = ((b_i)_{i\in I})$ then for all $i\in I$ there exist $z\in\mathbb{Z}, x_{1i}, x_{2i}, x_{si}, x'_{si}\in A_i$, such that

$$(a_i) = z(b_i) + (x_{1i})(b_i) + (b_i)(x_{2i}) + \sum\nolimits_{s\in J\,(finite)}(x_s i)(b_i)(x'_{si})$$

$$\text{Hence } \forall i\in I,\ \ a_i = zb_i + x_{1i}b_i + b_i x_{2i} + \sum\nolimits_{s\in J} x_{si}b_i x'_{si}$$

Implications b) and c) can be proved in a similar way as implication a) . While the last implication is a direct corollary of implications b) and c).

Proposition 2. *Let* A_i *for* $i\in I(\mathit{finite})$ *be s-unital rings and let* $A=\sum_{i\in I}A_i$ *then*

a) $(a_i)_{i\in I} J(b_i)_{i\in I} \Leftrightarrow a_i Jb_i, \forall i\in I,$

b) $(a_i)_{i\in I} R(b_i)_{i\in I} \Leftrightarrow a_i Rb_i, \forall i\in I,$

c) $(a_i)_{i\in I} L(b_i)_{i\in I} \Leftrightarrow a_i Lb_i, \forall i\in I,$

d) $(a_i)_{i\in I} D(b_i)_{i\in I} \Leftrightarrow a_i Db_i, \forall i\in I,$

e) $(a_i)_{i\in I} H(b_i)_{i\in I} \Leftrightarrow a_i Hb_i, \forall i\in I,$

Proof. We only prove here the first assertion because the other ones can be proved similarly.
If $(a_i)_{i\in I}\mathcal{J}(b_i)_{i\in I}$ then by using Proposition 1 we get that $a_i\mathcal{J}b_i, \forall i\in I$
For the converse. Let A_i be s-unital rings and $a_i Jb_i$ for all $i\in I,$ then elements b_i satisfy the following equality

$$b_i = \sum\nolimits_{k=1}^{n_i} x_{ki}a_i y_{ki}, \text{ for } x_{ki}, y_{ki}\in A_i \text{ and } n_i\in\mathbb{N}.$$

Since *I* has a finite number of elements then it exists a natural number $n = max\{n_i \mid i\in I\}$.

We denote $x'_{ki} = \begin{cases} x_{ki} & k\le n_i \\ 0 & k>n_i \end{cases}$, $y'_{ki} = \begin{cases} y_{ki} & k\le n_i \\ 0 & k>n_i \end{cases}$, $\forall i\in I$.

It is easy to prove that $b_i = \sum\nolimits_{k=1}^{n} x'_{ki}\, a_i y'_{ki}$, for all $i\in I$. So we have that $(b_i) = \sum\nolimits_{k=1}^{n}(x'_{ki})(a_i)(y'_{ki})$, hence $((b_i)_{i\in I}) \subseteq ((a_i)_{i\in I})$. By commuting b_i with a_i we get that $((a_i)_{i\in I}) \subseteq ((b_i)_{i\in I})$, consequently $(a_i)_{i\in I}\mathcal{J}(b_i)_{i\in I}$.
The assertion e) follows from assertion b) and c).

Corollary 1. *Let* A_i *be s-unital rings where I has a finite number of elements then*
$\mathcal{J}_{\sum_{i\in I}a_i} = \sum\mathcal{J}_{ai}, \mathcal{R}_{\sum_{i\in I}a_i} = \sum\mathcal{R}_{a_i}, \mathcal{L}_{\sum_{i\in I}a_i} = \sum\mathcal{L}_{ai}, \mathcal{D}_{\sum_{i\in I}a_i} = \sum\mathcal{D}_{ai}, \mathcal{H}_{\sum_{i\in I}a_i} = \sum\mathcal{H}_{ai}$

Proof. Let $\sum x_i$ be an arbitrary element of $\mathcal{J}_{\sum_{i\in I}a_i}$. Then we have that $\sum x_i J\sum a_i$. By using Proposition 2 we get that $x_i Ja_i$ for all $i\in I$, thus $\sum x_i \in \sum J_{ai}$, hence $J_{\sum_{i\in I}a_i} \subseteq \sum J_{ai}$
Conversely. Let $\sum x_i$ be an arbitrary element of $\sum J_{a_i}$. So we have that for all $i\in I$, $x_i\in J_{a_i}$. Which imply that $(\sum x_i)J(\sum a_i)$. Thus $(\sum x_i)\in\mathcal{J}_{\sum a_i}$.

Proposition 3. *Let* A_i *be left (right) s-unital rings for all* $i \in I$ *and let* $(a_i)_{i\in I}, (b_i)_{i\in I} \in \prod_{i\in I} A_i$ *then*

$$(a_i)_{i\in I}\,\mathcal{L}(b_i)_{i\in I} \Leftrightarrow a_i \mathcal{L} b_i \forall i \in I \ ,$$
$$(\ (a_i)_{i\in I}\,\mathcal{R}(b_i)_{i\in I} \Leftrightarrow a_i \mathcal{R} b_i \forall i \in I \ ).$$

Proof : We only prove the case for the left s-unital rings because the other case can be prove analogusly. Let $(a_i)_{i\in I}\mathcal{L}(b_i)_{i\in I}$ then from Proposition 1 it follows that $a_i\mathcal{L}b_i$ for all $i \in I$. Conversely, let $a_i\mathcal{L}b_i$ for all $i \in I$, we have that $a_i = x_i b_i$ where $x_i \in A_i$, since A_i is a left s-unital ring. Hence we get that $(a_i)_{i\in I} = (x_i)_{i\in I}(b_i)_{i\in I}$, and $((a_i)_{i\in I})_l \subseteq ((b_i)_{i\in I})_l$. Similarly we can prove the other inclusion. Finally we get that $(a_i)_{li\in I}\,\mathcal{L}(b_i)_{li\in I}$.

Corollary 2. *Let* A_i *be left (right) s-unital rings for all* $i \in I$ *. Then* $\mathcal{L}_{\prod_{i\in I} a_i} = \prod \mathcal{L}_{a_i}$, *(* $\mathcal{R}_{\prod_{i\in I} a_i} = \prod \mathcal{R}_{a_i}$ *)*

Proposition 4. *Let* A_i *be s-unital rings for all* $i \in I$ *and let* $(a_i)_{i\in I}, (b_i)_{i\in I} \in \prod_{i\in I} A_i$ *then*

a) $(a_i)_{i\in I}\,\mathcal{D}(b_i)_{i\in I} \Leftrightarrow a_i \mathcal{D} b_i \forall i \in I$)

b) $(a_i)_{i\in I}\,\mathcal{H}(b_i)_{i\in I} \Leftrightarrow a_i \mathcal{H} b_i \forall i \in I$

Proof. Let $(a_i)_{i\in I}\mathcal{D}(b_i)_{i\in I}$ then it exists $(c_i)_{i\in I} \in \prod_{i\in I} A_i$ such that $(a_i)_{i\in I}\mathcal{L}(c_i)_{i\in I}$ and $(c_i)_{i\in I}\mathcal{R}(b_i)_{i\in I}$, hence by using Proposition 3 we get that $a_i\mathcal{L}c_i$ and $c_i\mathcal{R}b_i$, thus $a_i\mathcal{D}b_i$, for all $i \in I$.
Conversely. Let $a_i\mathcal{D}b_i$ for all $i \in I$. Then it exists $c_i \in A_i$ such that $(a_i\mathcal{L}c_i)$ and $(c_i\mathcal{R}b_i)$, since A_i are s-unital rings, then from proposition 3 we have that $(a_i)_{i\in I}\mathcal{L}(c_i)_{i\in I}$ and $(c_i)_{i\in I}\mathcal{R}(b_i)_{i\in I}$ hence $(a_i)_{i\in I}\mathcal{D}(b_i)_{i\in I}$.

Corollary 3. *Let* A_i *be s-unital rings for all* $i \in I$ *and let* $(a_i)_{i\in I} \in \prod_{i\in I} A_i$ *then* $D_{\prod_{i\in I} a_i} = \prod D_{a_i}$,

[3] S-unital semigroups are defined similarly as in rings.

Leme 3 *Let* S_i *be s-unital semigroups and* $(a_i)_{i\in I}, (b_i)_{i\in I} \in \prod_{i\in I} S_i$ *then*

$\mathcal{J}_{\prod_{i\in I} a_i} = \prod \mathcal{J}_{ai}, \mathcal{R}_{\prod_{i\in I} a_i} = \prod \mathcal{R}_{ai}, \mathcal{L}_{\prod_{i\in I} a_i} = \prod \mathcal{L}_{ai}, \mathcal{H}_{\prod_{i\in I} a_i} = \prod \mathcal{H}_{ai}, \mathcal{D}_{\prod_{i\in I} a_i} = \prod \mathcal{D}_{ai}$

Proof: Let $S = \prod_{i\in I} S_i$. Since for all $i \in I$, S_i are s-unital it is easy to prove that for all $b_i \in S_i$, $J(b_i) = S_i b_i S_i$.

If for all $i \in I$, $b_i \mathcal{J} a_i$ we have that $a_i = y_i b_i x_i$ where $y_i, x_i \in S_i$, hence

$(a_i)_{i\in I} = (y_i)_{i\in I}(b_i)_{i\in I}(x_i)_{i\in I}$

$\mathcal{J}_{(a_i)_{i\in I}} \le \mathcal{J}_{(b_i)_{i\in I}}$ thus $\mathcal{J}_{(b_i)_{i\in I}} = \mathcal{J}_{(a_i)_{i\in I}}$.

Conversely. If $(b_i)_{i\in I}\mathcal{J}(a_i)_{i\in I}$ then $(b_i)_{i\in I} = (y_i)_{i\in I}(a_i)_{i\in I}(x_i)_{i\in I}$ for $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} S_i$. Thus for all $i \in I$ we have that $b_i = y_i a_i x_i$ and $\mathcal{J}_{b_i} \le \mathcal{J}_{a_i}$, hence $\mathcal{J}_{b_i} = \mathcal{J}_{a_i}$.

The proof is the same for the other relations $\mathcal{R}, \mathcal{L}, \mathcal{H}, \mathcal{D}$.

Proposition 5. *Let* A_i *for* $i \in I$ *be s-unital rings, where I is a finite set. Then the relations* $\mathcal{J}, \mathcal{R}, \mathcal{L}, \mathcal{H}, \mathcal{D}$ *in every ring* A_i *coincide with Green's relations* $\mathcal{J}, \mathcal{R}, \mathcal{L}, \mathcal{H}, \mathcal{D}$ *in their respective multiplicative semigroup if and only Green's relations in the ring* $A = \sum_{i \in I} A_i$ *coincide with Green's relations in the multiplicative semigroup* $(A,.)$

Proof : We first prove that $\mathcal{J} \leq \mathcal{J}(.)$.

Let $(\sum a_i) J (\sum b_i)$ then $a_i J b_i$, hence $a_i J(.) b_i$, by using the above lemma we get that $(\sum a_i) J(.) (\sum b_i)$.
Thus $J = J(.)$.

For the converse we have to prove that for all $i \in I$, $J_i = J_i(.)$. Let $a_i J_i b_i$ for some $i \in I$. We denote $(\overline{a_i})$ the element of A such that the i-th coordinate is equal to a_i and everywhere else is zero. So, the following implications hold $(\overline{a_i}) J (\overline{b_i}) \Rightarrow (\overline{a_i}) J(.) (\overline{b_i}) \Rightarrow a_i J(.) b_i$

Thus $J_i = J_i(.)$.

References

[1] Rowen, Louis. *Ring Theory. Student edition*. Academic Press, Inc., Boston, MA, 1991
[2] Petro, P. *Green's Relations and Minimal Quasi-Ideals in Rings*. Communications in Algebra, 2002
[3] F, Cela, P,Petro. *On Green's relations in rings and their multiplicative semigroups*. Journal of natural and sciences and mathematics of UT, Vol 2. 2017
[4] H Tominaga . *On s-unital ring*. Mathematical Journal of Okayama, 1976
[5] J, Howie. *Fundamentals of Semigroup Theory*. London Mathematical Society Monographs. New Series, 12. Oxford Science Publications
[6] Steinfeld, Otto. *Quasi-ideals in rings and semigroups,* Akadémiai Kiadó, Budapest, 1978.