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Relation between associative ternary operations and neutral sequences

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Abstract

In this paper we show relation between associative ternary operations and neutral sequences. Hence, for the operation $[\ ]$ we say that is associative ternary operation if for $x_1, x_2, x_3, x_4, x_5 \in G$ hold $[[x_1x_2x_3]x_4x_5] = [x_1[x_2x_3x_4]x_5] = [x_1x_2[x_3x_4x_5]]$. The sequence e_1^{n-1} is called *i*-neutral for the operation $[\ ]$ where $i \in \{1,2,...,n\}$, if $[e_1^{i-1}xe_i^{n-1}] = x,\ \forall x \in G$. The element $e \in G$ is called *i*-neutral for the operation $[\ ]$ where $i \in \{1,2,...,n\}$ if the sequence $\overset{n-1}{e}$ is *i*-neutral for $[\ ]$. In this case *e* is called identity for the operation $[\ ]$. If $[\ ]$ is associative ternary operation in the set *G* and (e_1e_2) is 2-neutral sequence with the operation $[\ ]$ then we prove that (e_1e_2) and (e_2e_1) are *i*-neutral sequences with the operation $[\ ]$, $\forall i \in \{1,2,3\}$, e_1, e_2 are central elements for 3-groupoid $(G,[\ ])$, (f_1, f_2) is 1-neutral sequence if and only if (f_1, f_2) is 3-neutral sequence.
By the group of permutations we show that the condition of (e_1e_2) to be 2-neutral sequence, can't replace with the other condition of (e_1e_2) being 1-neutral or 3-neutral. By the properties of associative operation implies that if for ternary operation $[\ ]$ there is at last one associative primitive binary operation then also the operation $[\ ]$ is associative.

Keywords: Associative ternary operation, neutral sequence, 1-neutral sequence, 2-neutral sequence, 3-neutral sequence.

Introduction

In this part of the paper we will present some basic meanings, notes, and features related to algebraic structures that have only one n-arch operation. Here we give a generalized interpretation of relation in an nonempty set.
Let *G* be an nonempty set and *n* an positive integer. We know that every function f from G^n to *G* can be named as an *n*-arch operation on *G*, this means that for every *n*-arch $(x_1, x_2, ..., x_n)$ of elements from *G* with the operation *f* is accompanied by a single element of G that we write with $f(x_1, x_2, ..., x_n)$. Every *n*-arch operation is called finite operation. Especially when $n = 1,2,3$ then we have that the operation is unary, binary or ternary respectively.
Set *G* in order with the n-arch operation *f* is called *n* –arch groupoid or *n*-groupoid and we can write it with (G, f) or shortly with $\underline{G}$. The meaning of 2 –groupoid is equal with the meaning of regular groupoid. The unary groupoid will be called as monounary.
In the theory of n-arch groupids, are exploited the following notes.

The sequence of the elements $x_i, x_{i+1}, ..., x_j$ is noted by x_i^j. For $j < i$, that symbol represent the empty sequence, which in fact is the sequence with just one element. The number of all elements of a sequence is called the length of sequence. The length of an empty sequence is 0.

If for x_{i+1}^{i+t}, we have $x_{i+1} = x_{i+2} = ... = x_{i+t} = x$, then in the place of x_{i+1}^{i+t} we note $\overset{t}{x}$. From that the $f(x_1, x_2, ..., x_n)$ shortly can be noted by $f(x_1^n)$, where $f\left(x_1^i \overset{t}{x} x_{i+t+1}^n\right)$ represent the shortened form for $f\left(x_1, ..., x_i, \underbrace{x, ..., x}_{t}, x_{i+t+1}, ..., x_n\right)$.

That the above mentioned notes be more comfortable for use, in the place of the symbol f for the n-arch operation we can write [ ]. The above products will be noted in the following forms: $[x_1 x_2 ... x_n], [x_1^n], [x_1 ... x_i \underbrace{x ... x}_{t} x_{i+t+1} ... x_n]$ and $[x_1^i \overset{t}{x} x_{i+t+1}^n]$, respectively .

Example Let $G = \{2n+1 | n \in Z\}$ and note $[x_1 x_2 x_3] = x_1 + x_2 + x_3$, then $(G, [\ ])$ is 3-groupoid. The given set is not groupoid with respect to adding, because the add of two odd integer numbers is an even number, so the operation + is not closed on *G*.

1. Algebraic structures with a *n*-arch operation

For the ternary groupoid $(G, [\ ])$ we says that is taken from binary operation $\cdot$ if the relation $[xyz] = (x \cdot y) \cdot z$, for every $x, y, z \in G$. If $[xyz] = ((x \cdot y) \cdot z) \cdot b$, for every $x, y, z \in G$ and some fixed element $b \in G$ then for $(G, [\ ])$ we can say that is *b*-taken from $\cdot$.

If $(G, [\ ])$ is an *n*-groupoid, and *A* an nonempty subset of *G* with the condition $[a_1^n] \in A$ for every $a_1, a_2, ..., a_n \in A$, then we can say that $(A, [\ ]|_A)$ is an *n*- subgroupoid from $\underline{G}$.

2-subgroupoidi of 2-groupoidi is the subgroupoid from groupoid is subgroupoid from that groupoidi.

To define the mean of *n*-semigruop , firstly we will add the mean of composite product in the algebra (G, F), where *G* is an nonempty set and F the family of finite operations on *G*.

1. If $x \in G$ then we note $x = \Pi(x)$ where $\Pi(x)$ is composite product of order zero.

2. If $x, x_1, x_2, ..., x_n \in G$, f an n-arch operation from F and if $x = f(x_1^n)$, then we note $x = \Pi(x_1, x_2, ..., x_n)$. For the right part of the equation we can say that it is composite product of first order.

3. Suppose $y_i = \Pi_i(x_{i1}, x_{i2}, ..., x_{ik_i})$ is composite product of order r_i , $i = 1, ..., m$ and g is an m - arch operation from F. Then for the right part of the operation will be: $g(y_1, ..., y_m) = g(\Pi_1(x_{11}, x_{12}, ..., x_{1k_1}), ..., \Pi_m(x_{m1}, x_{m2}, ..., x_{mk_m})$

We says that it is composite product of the order $r = 1 + r_1 + ... + r_m$. This product can be noted like $\Pi(x_{11}, ..., x_{1k_1}, ..., x_{m1}, x_{m2}, ..., x_{mk_m})$.

The composite product can also be defined in every n - groupoid $(G, [\ ])$. Note that if $\Pi(x_1, x_2, ..., x_m)$ is an finite product in the n -groupoid $(G, [\ ])$, then m has the form $m = k(n-1)+1$, where k is an nonnegative integer.

So for $k=0$ we have the product of order zero, $x=\Pi(x)$. For this m which is 1 we have $m=0\cdot(n-1)+1$ which means that holds the equation $m=k(n-1)+1$. Let $m>1$ the relation let be valid for composite products with length smaller than m.

So, if $\Pi(x_1,x_2,...,x_m)=[\Pi_1(x_1,..,x_{m1}),\Pi_2(x_{m_1+1},...,x_{m_1+m_2}),...,\Pi_n(...,x_{m_1+m_2+...+m_n})]$ we have $m=m_1+m_2+...+m_n$. From the inductive suppose we have that $m_i=k_i(n-1)+1$ which means $m=k_1(n-1)+1+...+k_n(n-1)+1=(k_1+...+k_n+1)(n-1)+1$.

Especially, with the n-arch operation [ ] of a set G we can define the nm-arch operations $[\ ]'_i$, where $m=2n-1,\ i\in\{1,...,n\}$ like below:

$$\left[x_1^{2n-1}\right]'_1=\left[\left[x_1^n\right]x_{n+1}^{2n-1}\right],\left[x_1^{2n-1}\right]'_2=\left[x_1\left[x_2^{n+1}\right]x_{n+2}^{2n-1}\right],...,\left[x_1^{2n-1}\right]'_n=\left[x_1^{n-1}\left[x_n^{2n-1}\right]\right]$$

The m-arch operations $[\ ]'_i$ are called continue products of n-arch operation [ ] which in overall case are different with each other. Those kinds of products of the n-arch operation [ ] can also be created for every $m=k(n-1)+1$ where $k\in G,\ k\geq 2$.

For example, $\left[x_1^{k(n-1)+1}\right]'=\left[\left[...\left[\left[x_1^n\right]x_{n+1}^{2n-1}\right]...\right]x_{(k-1)(n-1)+2}^{k(n-1)+1}\right]$ is a continue product of just an m-arch operation derived from [ ]. So, the n-arch operation is called primitive operation for the other m-arch operation $[\ ]'$.

For the n-groupoid $(G,[\ ])$ we say that is associative or just n - semigroup if [ ] is associative operation, so:

$$\left[\left[x_1^n\right]x_{n+1}^{2n-1}\right]=\left[x_1\left[x_2^{n+1}\right]x_{n+2}^{2n-1}\right]=...\left[x_1^{n-1}\left[x_n^{2n-1}\right]\right]$$

$$\left[x_1^{2n-1}\right]'_1=\left[x_1^{2n-1}\right]'_2=...=\left[x_1^{2n-1}\right]'_n$$

For the n-groupoid $(G,[\ ])$ we say that is commutative if

$$[x_1...x_n]=\left[x_{v_1}...x_{v_n}\right]$$

Holds for every $x_1,...,x_n\in G$ and all permutations $v_1,...,v_n$ of numbers $1,...,n$.

We can say thet $(G,[\ ])$ is reducible related with the i-th place or short i- reducible for fixed $i\in\{1,...,n\}$ if $[a_1...a_{i-1}xa_{i+1}...a_n]=[a_1...a_{i-1}ya_{i+1}...a_n]\Rightarrow x=y$.

If $i=1$, then $(G,[\ ])$ is reducible from the left, and, if $i=n$ then $(G,[\ ])$ is reducible from the right. And, if $(G,[\ ])$ is i-reducible for every $i\in\{1,...,n\}$, then we can say that is reducible.

The sequence e_1^{n-1} is named i-neutral for [ ] where $i\in\{1,...,n\}$, if $\left[e_1^{i-1}xe_i^{n-1}\right]=x,\forall x\in G$.

For $i=n$ he is left neutral sequence, and for $i=1$ right neutral sequence for the operation [ ]. The sequence e_1^{n-1} is neutral for [ ] if it is i-neutral for every $i\in\{1,...,n\}$.

Element $e\in G$ is i-neutral for the operation [ ] where $i\in\{1,...,n\}$ if the sequence $\overset{n-1}{e}$ is i-neutral for [ ]. Element e is neutral related with operation [ ] if it is i-neutral for every $i\in\{1,...,n\}$. In that case e is also neutral element for [ ]. Element 1-neutral also is called left neutral, and the element n-neutral is also called right neutral. The set of all i-neutral elements related to the operation [ ] is noted by E_i where $i\in\{1,...,n\}$.

Element $c\in G$ is central element for n- groupoid $(G,[\ ])$ if for every $x_1,...,x_{n-1}\in G$, $\forall i,j\in\{1,2,..,n\}$ holds the relation

$$\left[x_1^{i-1}c\,x_i^{n-1}\right]=\left[x_1^{j-1}c\,x_j^{n-1}\right]\ .$$

The set of all central elements for the n-groupoid noted with $(G,[\ ])$ is the center of it and will be noted by $Z(G)$.

The element $x \in G$ will be idempotent if $[\overset{n}{x}] = x$. One n- groupoid where all the elements are idempotent is an idempotent n-groupoid.

2. Some *n*-semigroups. Examples

The n-arch operation [ ] on G is (i, j)-associative, where $i, j \in \{1, 2, ..., n\}$ and $i \neq j$ if for every $x_1, x_{2,}..., x_{2n-1} \in G$ holds the equation:

$$\left[x_1^{i-1}\left[x_i^{i+n-1}\right]x_{i+n}^{2n-1}\right] = \left[x_1^{j-1}\left[x_j^{j+n-1}\right]x_{j+n}^{2n-1}\right].$$

If the above mentioned relation holds for every $x_1, x_{2,}..., x_{2n-1} \in G$ and $i, j \in \{1, 2, ..., n\}$, then we can say that the operation [ ] is associative. In this case the n-groupoid $(G, [\ ])$ is n- semigroup.

It is clear that 2-semigroups are just semigroups. Sometimes we replace the note of structure $(S, *)$ with just S, and the product of elements of S can be noted with just writing the elements near to each-other.
With the theorem bellow we show that holds the associative rule in the class of n –semigroups.

Theorem 2.1. If $(G, [\ ])$ is an n-semigroup and $\Pi'(x_1, ..., x_m), \Pi''(x_1, ..., x_m)$ are two composite products, then $\Pi'(a_1, ..., a_m) = \Pi''(a_1, ..., a_m), \forall a_1, ..., a_m \in G$.

Proof. Above we showed that $m = k(n-1)+1$, where k is an nonnegative integer. For $k = 0$ and $k = 1$ the proposition holds because in two showed cases exist just one composite product with length $m = 1$ (of order zero), respectively just one composite product with length $m = n$ (of first order). The accuracy of the proposition for $k = 2$ flows from the definition of n-semigroups. Let suppose $k > 2$ and the proposition holds for product with length smaller than m, respectably $\Pi^*(a_1, ..., a_q) = [a_1 ... a_q]$ for every composite product Π^* with length $q = r(n-1)+1$ where $r > k$. Note $\left[x_1^m\right]' = \left[\left[x_1^n\right]x_{n+1}^m\right]$ (for $m = 0$, $[x]' = [x]$). If $\Pi(x_1, ..., x_m)$ is defined with

$$\Pi(x_1, x_2, ..., x_m) = [\Pi_1(x_1, ..., x_{m1}), \Pi_2(x_{m_1+1}, ..., x_{m_1+m_2}), ..., \Pi_n(..., x_{m_1+m_2+...+m_n})]$$

Then we have $\Pi(x_1, x_2, ..., x_m) = [[x_1^{m_1}][x_{m_1+1}^{m_1+m_2}]...[x_{m_1+m_2+...+m_{n-1}+1}^{m}]]$. Let p be the smallest number for every $m_{p+1} \geq n$. If in the right part of the last relation we note $y = \left[x_{p+1}^{p+n}\right]$ and apply the inductive supposition then:

$$\Pi(x_1, x_2, ..., x_m) = [x_1^p y x_{p+n+1}^m] = [[x_1^p y x_{p+n+1}^{2n-1}]x_{2n}^m] = [[x_1^{2n-1}]x_{2n}^m] = [x_1^m]'. \ \square$$

Like an consequence of this theorem we have:

Consequence 2.1.
(*i*) If $(G, [\ ])$ is an n-semigroup, then $(G, [\ ])$ can be considered like an $k(n-1)+1$-semigroup for every natural number k. ($[x_1^m] = [x_1^j [x_{j+1}^{j+r}] x_{j+r+1}^m]$, where $r = t(n-1)+1, t \in \square, t < k$, and any $j \in \{0, 1, 2, ..., m-r\}$).
(*ii*) If $(S, \cdot)$ is an semigroup and $[\ ]: S^n \to S$ is an n-arch operation on S defined by $\left[x_1^n\right] = x_1 \cdot x_2 \cdot ... \cdot x_n$, then $(S, [\ ])$ is an n-semigroup.

Proof.
(*i*) Let $(G, [\ ])$ be an n-semigroup and k an natural number. Let $m = k(n-1)+1$, $x_1, x_2, ..., x_{2m-1} \in G$, then $2m-1 = 2k(n-1)+1$, so from the theorem 2.1. exists just one (2m-1)-arch operation from the operation [ ]. We will note that one with the symbol [ ]. From that we have:

$\left[\left[x_1^m\right]x_{m+1}^{2m-1}\right]=\left[x_1\left[x_2^{m+1}\right]x_{m+2}^{2m-1}\right]=...=\left[x_1^{m-1}\left[x_m^{2m-1}\right]\right]$.

$(G,[\ ])$ can be considered like an *m*-semigroup, or an $(k(n-1)+1)$ –semigroup.

(*ii*) Let $(S,\cdot)$ be an semigroup and $[\ ]:S^n\to S$ an *n*-arch operation on S defined by $\left[x_1^n\right]=x_1\cdot x_2\cdot...\cdot x_n$. It is clear that the structure $(S,[\ ])$ is an *n*-groupoid. $(S,\cdot)$ is semigroup so from that for every $x_1,x_2,...,x_{2n-1}\in G$

$$(x_1...x_n)x_{n+1}...x_{2n-1}=$$

$$=x_1(x_2...x_{n+1})x_{n+2}...x_{2n-1}=...=x_1...x_{n-1}(x_n...x_{2n-1})$$

From this $\left[\left[x_1^n\right]x_{n+1}^{2n-1}\right]=\left[x_1\left[x_2^{n+1}\right]x_{n+2}^{2n-1}\right]=...=\left[x_1^{n-1}\left[x_n^{2n-1}\right]\right]$, $\forall x_1,...,x_{2n-1}\in G$ and $(S,[\ ])$ is an *n*-semigroup. □

From Consequence 1.3.1, can we construct an semigroup $(G,\cdot)$ from every *n*-semigroup $(G,[\ ])$, with the condition that for every $x_1,x_2,...,x_n\in G$ is true the relation $\left[x_1^n\right]=x_1\cdot x_2\cdot...\cdot x_n$. In general case we have negative answer, which is shown in the following example.

Example 2.1. Let $G=\{2k+1\,|\,k\in\square\}$ be the set of all odd integer numbers. We define the 5-arch operation [ ] on *G* with the relation $\left[x_1^5\right]=\sum_{i=1}^{5}x_i$.

Because of $x_i=2k_i+1, k_i\in\square, i\in\{1,2,3,4,5\}$ we have $\left[x_1^5\right]=\sum_{i=1}^{5}x_i=\sum_{i=1}^{5}(2k_i+1)=2k+1$, from what $(G,[\ ])$ is an 5-groupoid. Let suppose that exist $(G,\cdot)$ with the valid equality $\left[x_1^5\right]=x_1\cdot x_2\cdot x_3\cdot x_4\cdot x_5$, $\forall x_1,x_2,x_3,x_4,x_5\in G$, then for example:

$$9=1+1+1+1+1+1+1+1+1=((1+1+1+1+1)+1+1+1+1)=$$

$$=[[\overset{5}{1}]\overset{4}{1}]=((1\cdot1\cdot1\cdot1\cdot1)\cdot1\cdot1\cdot1\cdot1)=1\cdot1\cdot1\cdot1\cdot1\cdot1\cdot1\cdot1\cdot1=$$

$$=(1\cdot1)\cdot(1\cdot1)\cdot(1\cdot1)\cdot(1\cdot1)\cdot1=[\overset{4}{(1\cdot1)}1]=4(1\cdot1)+1$$

And this is true if $1\cdot1=2$ but $2\notin G$. In this case we will define the cover semigroup. Let *G* be an nonempty set and *F* the set of all finite nonempty sequences $a_1a_2...a_n$ of the elements of *G*. So the two sequences $a_1a_2...a_n$ and $b_1b_2...b_m$ are equal to each other if and only if $n=m\wedge a_i=b_i,\ i=1,2,...,n$. Let define the concatenation operation on *F* with:

$(a_1a_2...a_n)(b_1b_2...b_m)=a_1a_2...a_nb_1b_2...b_m$.

Set *F* related with the above defined operation in semigroup. It also is called free generated semigroup from *G*. Note that *F* is *n*-semigroup related with the operation defined in the case (*ii*).

The semigroup $(S,\cdot)$ is covered semigroup for the *n*-semigoup $(G,[\ ])$ if and only if it meets the following conditions:

(i) $G\subseteq S$;

(ii) every $a\in S$ is an product of the form $a=a_1a_2...a_n$, $a_1a_2...a_n\in G$.

(iii) $[\overset{n}{a_1}]=a_1a_2...a_n$, $\forall a_1a_2...a_n\in G$.

So, for example the semigroup (*G*,+) is covered semigroup of 2-semigroup with elements odd integer numbers related with the operation of addition.

Theorem 2.2. Let $(G,[\ ])$ be an *n*-semigroup and *F* an free semigroup generated from *G*. Let define the relation α of *F* with:

$u\alpha v \Leftrightarrow u = a_1...a_{i-1}a_ia_{i+1}...a_k$, $v = a_1...a_{i-1}b_1...b_na_{i+1}...a_k$

where $a_i = [b_1^n]$ is on $(G,[\ ])$. Let $\beta = \Delta \cup \alpha \cup \alpha^{-1}$, $a\gamma$ is transitive extension of β. Then:

(i) γ is congruence of semigroup *F*.

(ii) If $a_1a_2...a_k \in G$, then $a\gamma b = a_1...a_k$ in F if and only if $b = [a_1^k]$ in $(G,[\ ])$.

(iii) If $1 \le i \le j < n,\ a_1,...a_i,b_1,...b_j \in G$, then $a_1...a_i\gamma b_1...b_j \Rightarrow i = j$.

(iv) If $G^\gamma = \{a^\gamma \,|\, a \in G\}$, then G^γ is *n*- subsemigroup from semigroup $S = F/\gamma$ where the function $\psi : a \mapsto a^\gamma$ is isomorphism from $(G,[\ ])$ to $(G^\gamma,[\ ])$.

Proof.

(i) The relation α is reflexive because for every $a \in G, [a] = a$ which means that γ is also a reflexive relation, because it is symmetric and transitive means that it is relation of equivalence on *F*. It is easy to show that if $x\gamma y$, then $xz\,\gamma\, yz \wedge zx\,\gamma zy$ from where γ is congruence on *F*.

(ii) Let $a, a_1...a_k \in G$ the opposite of the proposition is clear. If $a = [a_1^k]$ on $(G,[\ ])$ from the reflexivity of α follows $a\alpha a_1...a_k$ on *F*, which means that $a\gamma a_1...a_k$ on *F*.

Let $a \in G$ and $a\alpha u$. From the definition of α follows that *u* has the form $u = b_1...b_n$ where $a = [b_1^n]$ on $(G,[\ ])$. If $u\alpha v$ then $v = b_1...b_{i-1}c_1...c_nb_{i+1}...b_k$, where $b_i = [c_1^n]$ is on $(G,[\ ])$ which means that $a = \left[b_1^{i-1}\left[c_1^n\right]b_{i+1}^n\right] = \left[b_1^{i-1}c_1^nb_{i+1}^n\right]$. During the continue of the process we have that $a \in G, a\alpha u_1, u_1\alpha u_2, ..., u_{k-1}\alpha u_k$ from which follows $u_k = a_1...a_{k(n-1)+1}$, where $a = \left[a_1^{k(n-1)+1}\right]$ is on $(G,[\ ])$. From this and from the definition of γ follow $a,b \in G \wedge a\gamma b \Rightarrow a = b$.

(iii) Let $u = a_1...a_i,\ a_1,..a_i \in G$ note $|u| = i$. From the definition of α is clear that $u\alpha v \Rightarrow |u| \equiv |v| (\mathrm{mod}(n-1))$ from which follows that $u\gamma v \Rightarrow |u| \equiv |v| (\mathrm{mod}(n-1))$. From $a_1...a_i = u\,\gamma\, v = b_1...b_j, 1 \le i \le j \le n-1$ then $i \equiv j (\mathrm{mod}(n-1))$, which holds just for $i = j$.

(iv) Let $a_1^\gamma,...,a_n^\gamma \in G^\gamma$. If $a = [a_1^n]$ is on $(G,[\ ])$ Then $a\alpha a_1...a_n$ from which follows that $a_1^\gamma,...,a_n^\gamma = a^\gamma$ is on G^γ, respectively G^γ is an *n*- subsemigroup from F/γ and ψ is an epimorphism from $(G,[\ ])$ on $(G^\gamma,[\ ])$. fom (ii) follows that ψ is an injective function.

The proposition (iv) provides that the *n*-semigroup $(G,[\ ])$ can be considered like an *n*-semigrup from $S = F/\gamma$. In this case S will be cover on $(G,[\ ])$. Note $S = G \cup G^2 \cup ... \cup G^{n-1}$ where $G^i \cap G^j = \phi, i \ne j \wedge a_1...a_i = a_1'...a_i'$ on $S \Leftrightarrow a_1...a_i\,\gamma\, a_1'...a_i'$ on *F*.

From all this we can conclude that for every *n* –semigroup exists a cover semigroup.

Proposition 2.1. The *n*-semigroup $(G,[\ ])$ has an 1-neutral sequence and an *n*-neutral sequence if and only if her free cover S has an neutral element.

Proof. Let $(G,[\ ])$ be an *n*-semigroup, and (e_1^{n-1}) an 1-neutral and *n*-neutral sequence. If S is a free cover of $(G,[\ ])$. Then for $e = e_1...e_{n-1} \in S$ and $\forall x = x_1...x_i \in S$ are valid the relations:

$$xe = x_1...x_i e_1...e_{n-1} = x_1...x_{n-1}[x_i e_1^{n-1}] = x_1...x_{i-1}x_i = x$$

$$ex = e_1...e_{n-1}x_1...x_i = [e_1^{n-1}x_1]x_2...x_i = x_1x_2...x_i = x$$

Which means that *e* is neutral element of *S*. Otherwise, let *e* be neutral element of *S* .

Because of $S = G \cup G^2 \cup ... \cup G^{n-1}$ we have that $e \in G^i$ for every $i \in \{1,...,n-1\}$. Let $e \in G^p$ and $x \in G^{n-p}$. Then $e = e_1...e_p, x = x_1...x_{n-p}$ where $e_1,...,e_p, x_1,...,x_{n-p} \in G$ and

$$x = x_1...x_{n-p} = x_1...x_{n-p}e = x_1...x_{n-p}e_1...e_p \in G^n \subseteq G$$

From what we have $G^{n-p} \subseteq G$.

From the definition of the free cover S follows $G \cap G^j = \phi, \forall j \in 2,...,n-1$ from where we have $n-p=1 \Rightarrow p = n-1$. From that $a \in G$ we have

$$ae = ae_1...e_{n-1} = [ae_1^{n-1}] = a \text{ and } ea = e_1...e_{n-1}a = [e_1^{n-1}a] = a$$

Which means that the sequence (e_1^n) is 1-neutral and *n*-neutral for $(G,[\ ])$.□

Proposition 2.2. If the *n*-semigroup $(G,[\ ])$ has the cover *S* which is an reducible semigroup then and the other structure $(G,[\ ])$ is a reducible *n*-semigroup.

Proof. Let S be an reducible semigroup wich is cover of $(G,[\ ])$. If we use those conditions of S for every $i \in \{1,2,...,n\}$ we have:

$$\left[a_1^{i-1}xa_{i+1}^n\right] = \left[a_1^{i-1}ya_{i+1}^n\right] \Leftrightarrow a_1...a_{i-1}xa_{i+1}...a_n = a_1...a_{i-1}ya_{i+1}...a_n \Leftrightarrow$$

$$a_2...a_{i-1}xa_{i+1}...a_n = a_2...a_{i-1}ya_{i+1}...a_n \Leftrightarrow ... \Leftrightarrow xa_{i+1}...a_n = ya_{i+1}...a_n \Leftrightarrow$$

$$\Leftrightarrow xa_{i+1}...a_{n-1} = ya_{i+1}...a_{n-1} \Leftrightarrow ... \Leftrightarrow xa_{i+1} = ya_{i+1} \Leftrightarrow x = y.$$

From that the *n*-semigroup $(G,[\ ])$ is *i* –reducible for every $i \in \{1,2,...,n\}$, so $(G,[\ ])$ is an reducible *n*-semigroup. □

Know that in every semigroup $(S,\cdot)$ we can put the neutral element $e \notin S$, such as the structure $(S \cup \{e\},\circ)$ is a semiroup which contains $(S,\cdot)$ like its subsemigroup. Is enough to define the operation $\circ$ from $\cdot$ noting $x \circ y = x \cdot y,\ \forall x,y \in S,\ e \circ e = e \wedge x \circ e = e \circ x = x, \forall x \in S$.

Also, has been proof that it can be founded an analogue construction for every *n*-semigroup.

Proposition 2.3. Let $(G,[\ ])$ be an *n*-semigroup. Then $(G,[\ ])$ has a neutral element if and only if there exists the semigroup $(G,\circ)$ with neutral element of the form

$$\left[x_1^n\right] = x_1 \circ x_2 \circ ... \circ x_n, \forall x_1,x_2,...,x_n \in G.$$

Proof. Let $(G,[\ ])$ be an *n*-semigroup with neutral element *e*. Let define the binary operation $\circ$ on *G* with $x \circ y = [xy\overset{n-2}{e}]$. Then

$$(x\circ y)\circ z=[[xy\overset{n-2}{e}]z\overset{n-2}{e}]=[xy[\overset{n-2}{e}ze]\overset{n-3}{e}]=$$

$$[xy[z\overset{n-1}{e}]\overset{n-3}{e}]=[x[yz\overset{n-2}{e}]\overset{n-2}{e}]=x\circ(y\circ z)$$

So $(G,\circ)$ is an semigroup. Also,

$$[x_1^n]=[[x_1^n]\overset{n-1}{e}]=[x_1[x_2^n e]\overset{n-2}{e}]=x_1\circ[x_2^n e]=x_1\circ[[x_2^n e]\overset{n-1}{e}]=$$

$$=x_1\circ[x_2[x_3^n ee]\overset{n-2}{e}]=x_1\circ x_2\circ[x_3^n ee]=\ldots=$$

$$=x_1\circ\ldots\circ x_{n-2}\circ[x_{n-1}^n\overset{n-2}{e}]=x_1\circ\ldots x_{n-2}\circ[[x_{n-1}^n\overset{n-2}{e}]\overset{n-1}{e}]=$$

$$=x_1\circ\ldots\circ x_{n-2}\circ[x_{n-1}[x_n\overset{n-1}{e}]\overset{n-2}{e}=x_1\circ\ldots\circ x_{n-2}\circ x_{n-1}\circ[x_n\overset{n-1}{e}]=$$

$$=x_1\circ\ldots\circ x_{n-2}\circ x_{n-1}\circ x_n$$

and $x\circ e=[x\overset{n-1}{e}]=x\wedge e\circ x=[ex\overset{n-2}{e}]=x$, then *e* is an neutral element of the semigroup $(G,\circ)$.

Otherwise, let $(G,\circ)$ be a semigroup with neutral element such as $[x_1^n]=x_0\circ x_2\circ\ldots\circ x_n$. Then for every $x\in G,\forall i\in\{1,2,\ldots n\}$ holds the relation

$$[\overset{i-1}{e}\,x\,\overset{n-i}{e}]=\underbrace{e\circ\ldots\circ e}_{i-1}\circ x\circ\underbrace{e\circ\ldots\circ e}_{n-i}=x$$

and *e* is an neutral element for the *n*- semigroup, $(G,[\ ])$.□

Proposition 2.4. Let $(G,[\ ])$ be an semigroup and let $\circ$ be an binary associative primitive operation for [ ]. Then the operation [ ] has neutral sequence if and only if it has nautral element.

Proof. Let e_1^{n-1} be an neutral sequence for the operation [ ]. Then holds the relation $e=e_1\circ\ldots\circ e_{n-1}\in G,\forall x\in G$ and from that we have

$$x\circ e=x\circ e_1\circ\ldots\circ e_{n-1}=[xe_1^{n-1}]=x$$
$$e\circ x=e_1\circ\ldots\circ e_{n-1}\circ x=[e_1^{n-1}x]=x$$

This means that the element *e* is neutral for $(G,\circ)$, so *e* is also neutral for $(G,[\ ])$.

Otherwise, let *e* be neutral for $(G,[\ ])$. Then $\overset{n-1}{e}$ is neutral sequence for [ ].□

Proposition 2.5. Let $(G,[\ ])$ be a *n*-semigroup and $\circ$ an binary associative primitive operation for [ ]. If the operation [ ] don't have any neutral element , then it doesn't have neutral sequence.

Theorem 2.3. Let $(G,[\ ])$ be an *n*-semigroup without neutral element. If there exist a binary associative primitive for [ ], then the *n*- semigroup $(G,[\ ])$ can be with neutral element.

3. Representation of associative ternary operation with neutral sequences

In this paragraph with [ ] we note the ternary associative operation on G. For [ ] we say that is a associative ternary operation if for every $x_1,x_2,x_3,x_4,x_5\in G$ is valid the following relation:

$$[[x_1x_2x_3]x_4x_5]=[x_1[x_2x_3x_4]x_5]=[x_1x_2[x_3x_4x_5]].$$

Based on the theorem 1.3.1 if [ ] is an associative operation then exist just one $(2k+1)$-arch operation $[\,]'$ excluded from [ ].

The range (e_1e_2) which is neutral 2-neutral is in fact *i*-neutral for $i \in \{1,2,3\}$, while e_1 dhe e_2 are key elements for 3-groupoid $\left(G,[\,]\right)$.

Theorem 3.1. Let [ ] be an associative ternary operation on G and (e_1e_2) an 2-neutral sequence related with operation [ ]. Then:

(i) (e_1e_2) and (e_2e_1) are *i*-neutral sequences related with the operation [ ] for every $i \in \{1,2,3\}$.

(ii) e_1 and e_2 are central elements for 3-groupoid $\left(G,[\,]\right)$.

(iii) the sequence (f_1f_2) is 1-neutral iff and only if (f_1f_2) is an 3-neutral sequence.

(iv) $[x\overset{2k}{e_1}]=y$ if and only if $[y\overset{2k}{e_2}]=x$.

Consequence 3.1. If [ ] is an associative operation and $E_2=\phi$, then $E_2 \subseteq E_1=E_3$.

Consequence 3.2. If $(G,\cdot)$ is an semogroup and if there exists $a,b \in G$ such as for every $x \in G$ holds the relation $axb=x$, then $ab=ba$ is an neutral element on the semigroup for every $x \in G$ we will have $ax=xa \wedge bx=xb$.

From the definition of the excluded operations we have that for every *n* –arch operation [ ], we can say that [ ] is primitive. This is the only *n*-arch primitive operation for [ ]. Let $[\,]'$ be an *n*-arch primitive operation for [ ]. Then we have $[x_1^n]=[[x_1^{m_1}]_1,[x_{m_1+1}^{m_1+m_2}]_2,...,[x_{m_1+m_2+...+m_{n-1}}^{m}]_n]'$ and from this $[\,]_1,[\,]_2,...,[\,]_n$ $m_1,m_2,...,m_n$ operations excluded from $[\,]'$, where $m_1+m_2+...+m_n=n$. From that we have $m_1,m_2,...,m_n \in G$ and $m_1=...=m_n=1$. So $[x]_i=x$, $\forall i=\{1,...,n\}$, or $[x_1^n]=[x_1^n]'$, respectively $[\,]=[\,]'$. From the theorem 1.3.1 follows that an unary operation can be excluded just one unary operation because $k(1-1)+1=1$. This means that if $n=3$ then the other primitive operations are binary operations.

Theorem 3.2. Let [ ] be an associative ternary operation on G from which the sequence (e_1e_2) is 1-neutral and let $[\,]'$ be just an binary operation. Then for every $x,y,z \in G$ follows $[x[yz]']'=[xyz]$ if and only if exist the element $e \in G$ such as the element *e* is 1-neutral for operation [ ] and for every $x,y \in G$ holds the relation $[xy]'=[xye]$.

Theorem 3.3. Let [ ] associative ternary operation on the set G from which (e_1e_2) is 3-neutral and let $[\,]'$ binary operation. Then from every $x,y,z \in G$ follows $[[xy]'z]'=[xyz]$ if and only if there exist the element $e \in G$ such as *e* is 3-neutral for the operation [ ] and for all $x,y \in G$ holds the relation $[xy]'=[exy]$.

Proof. For every $x,y,z \in G, [[xy]'z]'=[xyz] \Rightarrow x=[e_1e_2x]=[[e_1e_2]'x]'=[ex]'$, where

$e=[e_1e_2]' \Rightarrow [xy]'=[[ex]'y]'=[exy],\ \forall x,y \in G$

and from that follows $[eex]=[[ee]'x]'=[ex]'=x, \forall x \in G$. The element *e* is 3-neutral for the operation [ ]. Otherwise, let *e* be an 3-neutral element for the operation [ ] and for every $x,y \in G$ holds the relation le të $[xy]'=[exy]$. Also for every $x,y,z \in G$ are valid the relations $[[xy]'z]'=[e[exy]z]=[ee[xyz]]=[xyz]$. □

Conclusions

We know that every function f from G^n to G can be named as an n-arch operation on G, this means that for every n-arch $(x_1, x_2, ..., x_n)$ of elements from G with the operation f is accompanied by a single element of G that we write with $f(x_1, x_2, ..., x_n)$. Every n-arch operation is called finite operation. Especially when $n = 1,2,3$ then we have that the operation is unary, binary or ternary respectively. In the field of the finite operations in this paper we defined the ternary operation like $[\ ]$ like an associative ternary operation if for $x_1, x_2, x_3, x_4, x_5 \in G$ holds $[[x_1 x_2 x_3] x_4 x_5] = [x_1 [x_2 x_3 x_4] x_5] = [x_1 x_2 [x_3 x_4 x_5]]$. We also introduced the definition of neutral sequences by the sequence e_1^{n-1} which is called i-neutral for the operation $[\ ]$ and $i \in \{1,2,...,n\}$, if $[e_1^{i-1} x e_i^{n-1}] = x$, $\forall x \in G$. The element $e \in G$ was called i-neutral for the operation $[\ ]$ where $i \in \{1,2,...,n\}$ if the sequence e^{n-1} is i-neutral for $[\ ]$. In this case we showed that e is called identity for the operation $[\ ]$. If $[\ ]$ is associative ternary operation in the set G and $(e_1 e_2)$ is 2-neutral sequence with the operation $[\ ]$ then we prove that $(e_1 e_2)$ and $(e_2 e_1)$ are i-neutral sequences with the operation $[\ ]$, $\forall i \in \{1,2,3\}$, e_1, e_2 are central elements for 3-groupoid $(G, [\ ])$, (f_1, f_2) is 1-neutral sequence if and only if (f_1, f_2) is 3-neutral sequence.

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