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AFFINE AND PROJECTIVE PLANES CONSTRUCTED FROM RINGS

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Abstract

In this paper, firstly, we show that there can be constructed an affine plane $\mathrm{A}(F)$ from ternary ring in natural way. Firstly, we present the basic properties of affine and projective planes including their completion with each other, respectively, then we continue with their definition over a skew-field. Considering that not all affine planes are of the form$\mathrm{A}^2(F)$, we use the Desargues properties to characterize them. Mathematically projective geometry if even more natural than its affine version.
The work continues by obtaining the projective planes $\mathrm{P}(F)$ by "completing" the plane constructed from ternary system $(\square, F)$, by means of projective completion and then constructing affine planes from projective planes by means of affine restriction. One should add a new point "at infinity" for each direction, there will also be a line "at infinity". Affine lines ℓ are too short, we must force the projective line to contain the direction $\hat{\ell} = \ell \cup \{[\ell]\}$. The concepts are equivalent, if you have got one, you have got the other. In the end we show the process of affinization and projectivization of the projective and affine plane. Affinization of projectivization of an affine plane $\mathrm{A}(F)$ may depend on the choise of line removed from $\widehat{\mathrm{A}}(F)$, and need not be isomorfiphic to $\mathrm{A}(F)$.

Keywords: affine plane, projective planes, affinization, ternary ring.

1. Introduction

Definition 1.1. (Parallel). Two lines ℓ, m are parallel if either $\ell = m$ or there is no point incident to both:

$$\ell \parallel m \; iff \; [\ell = m] \vee [(\forall a)\neg(a \mathrm{I} \ell \wedge aIm)] \tag{1.2}$$

Notice that it captures the intuition of $\mathbb{R}^2$, not of $\mathbb{R}^3$. This notion plays a role only when studying affine planes and disappears when studying projective planes [2].

Definition 1.2. (Affine plane) [3, 5] An affine $\mathbb{A}$ is an incidence geometry satisfying the following axioms **AP$_1$**, **AP$_2$**, **AP$_3$**:

AP$_1$.$(\forall a)(\forall \mathrm{b})[(a \neq \mathrm{b}) \rightarrow (\exists! \ell)(a \mathrm{I} \ell \wedge bI\ell)]$.

AP$_2$. $(\forall a)(\forall \ell)(\exists! m)[(a \mathrm{I} m) \wedge (\ell \parallel m)]$.

AP$_3$. There exist three non-collinear points.

Definition 1.3. (Projective plane) [1, 3] A projective plane is an incidence geometry$(\mathcal{P}, \mathcal{L}, I)$satisfying thefollowing axioms **PP$_1$**, **PP$_2$**, **PP$_3$**:

PP$_1$. Through any two distinct points, there is a unique line.

PP$_2$.Any two distinct lines meet at a unique point.

PP$_3$.There are four points not three of which are collinear.

Definition 1.4. [5, 8] We say $(\mathbb{R}, \mathbb{F})$ is a *ternary ring* if it satisfies the axioms:

T1. For any $x, m, y \in \mathbb{R}$ the equation$F(x, m, \ell) = y$ has a unique solution ℓ.

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T2. For any $x, x', y, y' \in \mathbb{R}$ the pair of equations $F(x, m, \ell) = y$, $F(x', m, \ell) = y'$ has a unique solution m, ℓ if $x \neq x'$.

T3. For any $\ell, \ell', m, m' \in \mathbb{R}$ the equation $F(x, m, \ell) = F(x, m', \ell')$ has a unique solution x if $m \neq m'$.

Definition 1.5. [5] (Affine plane over a skew-field $\mathbb{A}^2(\mathbb{F})$). Let $\mathbb{F}$ be a skew-field. Let$\mathbb{A}^2(\mathbb{F})$ be the following incidence geometry:

- Points are pairs $(x, y) \in \mathbb{F}^2$;
- Lines are sets of them $\{p + t\vec{v} : t \in \mathbb{F}\}$ for p a point and $\vec{v} \neq \vec{o}$ a non-zero vector;
- Incidence is set-theoretic membership.

This generalizes the familiar case of $\mathbb{A}^2(\mathbb{R})$ as it allows skew-fields.

Definition 1.6. [2, 4] (Projective plane over a skew-field $\mathbb{P}^2(\mathbb{F})$). Let $\mathbb{F}$ be a skew-field. Let $\mathbb{P}^2(\mathbb{F})$ be the following incidence geometry:

- As points, the left-vector lines of $\mathbb{F}^3$;
- As points, the left-vector planes of $\mathbb{F}^3$;
- As incidence relation, set-theoretic inclusion $\subseteq$.

2. Planes From Rings

Let $\mathbb{R}$ be any set with a ternary composition $F: \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ (thinking of $F(x, m, \ell) = xm + \ell$)). An isomorphism $\mathbb{R} \to \mathbb{R}$ of ternary systems is a bijective map preserving with little algebraic structure we have, namely the ternary composition:

$$\sigma(F(x, m, \ell)) = F(\sigma(x), \sigma(m), \sigma(\ell)) \tag{2.1}$$

The plane associated with the ternary system ($\mathbb{R}$, F) is $\mathbb{A}^2(\mathbb{R}) = (\mathcal{P}, \mathcal{L}, I)$, where:

$$\begin{cases} F(\mathbb{R}, \mathbb{F}) = \mathbb{R} \times \mathbb{R} = \{\{x, y\} | x, y \in \mathbb{R}\} \\ \mathcal{L}(F, \mathbb{R}) = \mathbb{R} \cup \mathbb{R} \times \mathbb{R} = \{[a], [m, \ell] | a, m, \ell \in \mathbb{R}\} \\ I(\mathbb{R}, \mathbb{F}) \text{is defined by} (x, y)\ I\ [a] \Leftrightarrow x = a \\ (x, y)\ I\ [m, \ell] \Leftrightarrow y = F(x, m, \ell). \end{cases} \tag{2.2}$$

$\mathbb{A}^2(\mathbb{F})$ satisfies the Parallel Criterion,

$$[m, \ell] || [m', \ell'] \Leftrightarrow m = m' \tag{2.2}$$

which ensures that the parameter *m* measures slope [8, 9].

We consistently write $F(x, m, \ell) = y$ even though at first glance all the variables x, m, ℓ, y from $\mathbb{R}$ are on the same footing and could be denoted by any letter we choose, the reason is that *x*,*y* will be the *x* and *y* coordinates of points, *m* will be the slope of a line and ℓ they-intercept of a line.

Theorem 2.1. (Plane Construction Theorem) The plane $\mathbb{A}^2(\mathbb{R})$ constructed from a ternary system ($\mathbb{R}$, *F*) is an affine plane satisfying the Parallel Criterion 2.2 if ($\mathbb{R}$, *F*) is a ternary ring with at least 2 distinct elements [4, 6].

Proof: Consider the axiom $\mathbf{AP_1}$that there be a unique line through any two points $P = (x, y)$, $P' = (x', y')$. If $x = x'$ then there is a unique line $\mathcal{L}$ of the form $[a]$ incident to both points (namely $a = x = x'$), and no line of the form $[m, \ell]$ since $y = F(x, m, \ell)$, $y' = F(x', m, \ell)$ for $x = x'$ forces $y = y'$, $P = P'$ by single-valuedness of F. If $x \neq x'$ there is no $\mathcal{L} = [a]$ on both points, and the condition that there be a unique $\mathcal{L} = [m, \ell]$on both is that $y = F(x, m, \ell)$,$y' = F(x', m, \ell)$have a unique solution m, ℓ for $x \neq x'$. Thus, $\mathbf{AP_1}$ is equivalent to **T2**.

The axiom that two lines $\mathcal{L}$,$\mathcal{L}'$intersect in exactly 1 or 0 points involves three cases. If$\mathcal{L} = [a']$ for $a \neq a'$ the lines do not intersect, since if (x, y) were on $\mathcal{L}$ and $\mathcal{L}'$ we would have$a = x = a'$. If $\mathcal{L} = [a]$, $\mathcal{L}' = [m, \ell]$ then (x, y) is on $\mathcal{L}$ and $\mathcal{L}'$iff$x = a$ and $y = F(x, m, \ell)$, so the unique point of intersection is $(a, F(a, m, \ell))$. If $\mathcal{L} = [m, \ell]$, $L' = [m', \ell']$ then the points of intersection $P = (x, y)$ are the solutions of the equations $y = F(x, m, \ell) = F(x, m', \ell')$. The Parallel Criterion is that no solution exists (the lines are parallel) iff $m = m'$. Thus, $\mathbf{AP_2}$ and the Parallel

Criterion 2.2 together are equivalent to the condition that $F = (x, m, \ell) = F(x, m', \ell')$ have a unique solution if $m \neq m'$ and no solution if $m = m'$ $(\ell \neq \ell')$, ie. to T3 and the uniqueness part of **T1**.
The axiom **AP3** that through each $P = (x, y)$ there is a unique line $\mathcal{L}'$ parallel to $\mathcal{L}$ breaks into two cases. If $\mathcal{L} = [a]$ we saw the only lines parallel to $\mathcal{L}$ are the $\mathcal{L}' = [a']$, and there is only one of three, incident to P (namely $a' = x$). If $\mathcal{L} = [m, b]$ we saw the only lines parallel to $\mathcal{L}$ are the$\mathcal{L}' = [m, \ell']$, which is on (*x*,*y*) if $y = F(x, m, \ell')$, so **AP3** is equivalent to the existence of unique solutions ℓ' of equations $y = F(x, m, \ell')$. This is just **T1**.
Thus, we have a pro-affine plane if ($\mathbb{R}$, F) is a ternary ring. The axiom **AP4** that a 3-point exist amounts to the condition that $|\mathbb{R}| \geq 2$: if $\mathbb{R} = \{r\}$ contains only one element then the plane contains only one point (r,r), while if $\mathbb{R}$ contains $r \neq s$ the already (*r*, *r*), (*r*, *s*), (*s*, *s*) from a 3-points (they are not all on a common line $\mathcal{L}$ since if they are on $\mathcal{L}[a]$ we would have$r = a = s$, and if they are on $\mathcal{L} = [m, \ell]$ then $r = F(r, m, \ell) = s$).
This construction is functorial in the sense that any isomorphism $\sigma: \mathbb{R} \to \widetilde{\mathbb{R}}$ induces an isomorphism
$\mathbb{A}^2(\sigma)$: $\mathbb{A}^2$ ($\mathbb{R}$)$\to\mathbb{A}^2$ ($\widetilde{\mathbb{R}}$) defined by:

$$(x, y) \to (\sigma(x), \sigma(y))$$
$$[m, \ell] \to [\sigma(m), \sigma(\ell)]$$
$$[a] \to [\sigma(a)] \qquad (2.3)$$

$\mathbb{A}^2(\mathbb{R})$ is built up from $\mathbb{R}$ using only the ternary structure of $\mathbb{R}$, so any map preserving this ternary structure must also preserve the derived geometric structure [6, 7, 8]. For skeptics: $\mathbb{A}^2(\sigma)$ certainly is bijective from points to points and lines to lines and perverse incidence since $(x, y)\ I\ [a] \Leftrightarrow x = a \Leftrightarrow \sigma(x) = \sigma(a) \Leftrightarrow (\sigma(x), \sigma(y))\ I\ [\sigma(a)]$ and $(x, y)\ I[m, \ell] \Leftrightarrow F(x, m, \ell) = y \Leftrightarrow F(\sigma x, \sigma m, \sigma\ell) = \sigma(F(x, m, \ell)) = \sigma \Leftrightarrow (\sigma x, \sigma y)\ I[\sigma m, \sigma\ell]$. The other requirements for a functor are trivially met: if σ=1 is the identity map from $\mathbb{R}$ to $\mathbb{R}$ by its very definition $\mathbb{A}^2(\sigma) = 1$ is the identity map on $\mathbb{A}^2(\mathbb{R})$, and similarly if $\mathbb{R} \to \mathbb{R} \to \mathbb{R}$ then by definition$\mathbb{A}^2(\tau)\circ\mathbb{A}^2(\sigma) = \mathbb{A}^2\ (\tau\circ\sigma)$. Thus, we have a functor.
In short, we can construct affine planes in a natural way from ternary rings [8, 9].
We noticed before that we could construct ternary systems out of rings. Indeed, if $(\mathbb{R}, +, \cdot)$ consists of a set $\mathbb{R}$ with two binary operations + and · we can form a ternary composition $F(x, m, \ell) = x \cdot m + \ell$.
The conditions on $(\mathbb{R}, +, \cdot)$ in order that ($\mathbb{R}$, F) be ternary are:

R1. An equation $x \cdot m + \ell = y$ has unique solution ℓ

R2.A pair of equations $x \cdot m + \ell = y, x' \cdot m + \ell = y'$ for $x \neq x'$ have a unique solution m,ℓ

R3. An equation $x \cdot m + \ell = x \cdot m' + \ell$ for $m \neq m'$ has a unique solution x.

The construction of ($\mathbb{R}$, *F*) from $(\mathbb{R}, +, \cdot)$ is functorial. In the next section, we will investigate when $(\mathbb{R}, +, \cdot)$ can be recovered from ($\mathbb{R}$, *F*).

3. From Affine to Projectiveand Vice Versa

Another way to obtain projective planes is by 'completing' affine planes, a procedure we now describe. In the affine world, some intersections are missing: if $\ell \parallel m$are distinct, parallel affine lines, then we should add a 'point at infinity' where they meet at last. Now, if $\ell \parallel m \parallel n$ then the same point should be added for the missing intersection $\ell \cap m$ and the missing intersection $m \cap n$. So, one is not working with pairs of parallel lines, but with whole sets of pairwise parallel lines.
Lemma 3.1. Let $\mathbb{A}$be any affine plane. Then $\parallel$is an equivalence relation [2, 3].

Proof. This holds of $\mathbb{A}^2(\mathbb{F})$ for $\mathbb{F}$ a skew-field, as one has a detailed description of parallelism from the proof of Proposition 1.3.2. But we want a general proof, a proof using only axioms **AP₁**, **AP₂**, **AP₃**. Remember that in this abstract setting ℓ and m are parallel if $\ell = m$ or ℓ and m do not meet. We freely replace $\mathbb{A}$ by an isomorphic plane where incidence is given by membership. There are three things to check:

- **Reflexivity.** Every line ℓ satisfies $\ell = \ell$, so $\ell \parallel \ell$.
- **Symmetry.** If $\ell \parallel m$ then either $\ell = m$ or $\ell \cap m = \emptyset$; in either case $m \parallel \ell$.
- **Transitivity.** Suppose $\ell \parallel m$ and $m \parallel n$; we show that $\ell \parallel n$. If $\ell = n$ we are done. Otherwise, we must prove $\ell \cap n = \emptyset$ .So suppose not; by **AP₁** there is $a \in \ell \cap n$. Now by **AP₂**, there is a unique line parallel to m and containing a; however, this applies to both ℓ and n. So actually $\ell = n$, a contradiction showing that $\ell \cap n = \emptyset$. Hence $\ell \parallel n$, as desired.

For ℓ an affine line, let $[\ell]$ be its equivalence class and call it itsdirection.
Intuitively one should add a new point 'at infinity' for each direction; there will also be a line 'at infinity'. But ordinary, affine lines ℓ are now too short: we must force a projective line to contain the direction. This explains why we go through $\widehat{\ell}$ below [3].
Definition 3.1. (Projectivization $\widehat{\mathbb{A}}$ of an affine plane). Let $\mathbb{A} = (\mathcal{P}, \mathcal{L} \in)$ be an affine plane. The projectivization of A is the incidence geometry $\widehat{\mathbb{A}} = (\widehat{\mathcal{P}}\,, \hat{\mathcal{L}}, \in)$ defined as follows:

- For each $\ell \in \mathcal{L}$, let $\widehat{\ell} = \ell \cup \{[\ell]\}$ be the 'completion of line ℓ';
- Let $\ell_\infty = \{[\ell] : \ell \in \mathcal{L}\}$ be the 'line of directions';
- $\widehat{\mathcal{P}} = \mathcal{P} \cup \ell_\infty = \mathcal{P} \cup \{[\ell] : \ell \in \mathcal{L}\}$;
- $\hat{\mathcal{L}} = \{\hat{\ell} : \ell \in \mathcal{L}\} \cup \{\ell_\infty\}$.

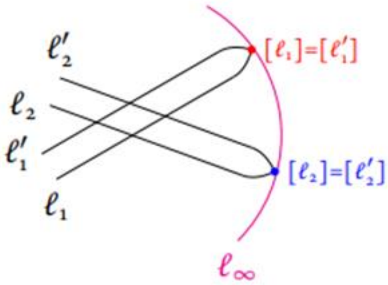

Figure 3.1. Completion of each affine line ℓ into $\widehat{\ell}$, without forgetting the line into infinity.

Proposition 3.1. Let $\mathbb{A}$ be an affine plane. Then $\widehat{\mathbb{A}}$ is a projective plane.
Proof: There are three axioms to check.
PP₁. Let $\alpha \neq \beta$ be two points in $\widehat{\mathcal{P}}$. We see several cases.

- If $\alpha = a \in \mathcal{P}$ and $\beta = b \in \mathcal{P}$ then by **AP₁** there is a unique line $\ell \in \mathcal{L}$ containing both. Clearly $\alpha, \beta \in \hat{\ell}$. We also have uniqueness. If another line $\lambda \in \hat{\mathcal{L}}$ contains α and β, then λ cannot be ℓ_∞, so $\lambda = \widehat{m}$, for some affine line $m \in \mathcal{L}$. Then $a, b \in m$, so by uniqueness in **AP₁** one has $m = \ell$ and therefore $\lambda = \widehat{m} = \hat{\ell}$. So, $\hat{\ell}$ is the only line of $\hat{\mathcal{L}}$ containing α and β.
- Suppose $\alpha = a \in \mathcal{P}$ but $\beta \in \widehat{\mathcal{P}} \setminus \mathcal{P} = \ell_\infty$ (the other case is similar). Then by definition, there is $\ell \in \mathcal{L}$ with $\beta = [\ell]$. Now by **AP₂** there is a unique $m \in \mathcal{L}$ with $a \in m$ and $m \parallel \ell$. Then on the one hand $\alpha \in \widehat{m}$, and on the other hand $[\ell] = [m] \in \widehat{m}$.Now to uniqueness. If a line λ contains a and β, then it cannot be ℓ_∞. So, it is of the form $\hat{n}$ for some affine line n. Now $\alpha \in \hat{n}$ implies $a \in n$, and $\beta = [\ell] \in \hat{n}$ implies $[\ell] = [n]$, that is, $\ell \parallel n$.
  By uniqueness in **AP₂** one has $n = m$, and therefore $\lambda = \hat{n} = \widehat{m}$.
- Now suppose $\alpha, \beta \in \ell_\infty$. Clearly ℓ_∞ is the only line in $\hat{\mathcal{L}}$ incident to both α and β.

PP2. Trivial
PP3. Obvious from **AP3** and the definition of the projectivization.
The converse operation of 'downgrading from projective to affine' involves choosing the line to be removed.

Definition 3.2. [5, 7] (Affinization $\breve{\mathbb{P}}_\lambda$ of a projective plane). Let, $\mathbb{P} = (\mathcal{P}, \mathcal{L}, \in)$ be a projective plane. Fix one line $\lambda \in \mathcal{L}$. The affinization of $\mathbb{P}$, with respect to λ, is the incidence geometry $\breve{\mathbb{P}}_\lambda = (\breve{P}, \breve{\mathcal{L}}, \in)$ defined as follows:

- For each $\mu \in \mathcal{L} \setminus \{\lambda\}$, set $\breve{\mu} = \mu \setminus \lambda$;
- $\breve{\mathcal{P}} = \mathcal{P} \setminus \lambda$;
- $\breve{\mathcal{L}} = \{\breve{\mu} : \mu \in \mathcal{L} \setminus \{\lambda\}\}$

Not all our notation reflects dependence on λ.

- The isomorphism type of $\breve{\mathbb{P}}_\lambda$ depends on the line λ you chose to remove. Affinization is not uniquely defined, one may not say 'the affinization' without specifying λ. (Cf. ''the projectivization', which is well-defined.)
- In particular, an arbitrary affinization of the projectivization of an affine plane $\mathbb{A}$ may depend on the choice of line removed from $\widehat{\mathbb{A}}$,and need not be isomorphicto $\mathbb{A}$.
- But, starting from projective $\mathbb{P}$ and letting $\mathbb{A} = \breve{\mathbb{P}}_\lambda$, one has $\widehat{\mathbb{A}} \simeq \mathbb{P}$ regardless of λ.

4. Projectivizing $\mathbb{A}_2(\mathbb{F})$, Affinizing $\mathbb{P}_2(\mathbb{F})$

Applying the projectivization/affinization procedures to $\mathbb{A}^2(\mathbb{F})$ and $P^2(\mathbb{F})$ gives what one expects [5].

Proposition 4.1. Let $\mathbb{F}$ be a skew-field. Then:

(i) $\widehat{\mathbb{A}^2(\mathbb{F})} \simeq \mathbb{P}^2(\mathbb{F})$;

(ii) For any line, $\breve{\mathbb{P}}^2(\mathbb{F}) \simeq \mathbb{A}^2(\mathbb{F})$.

Proof.

(i) We describe an isomorphism. Let $H_0 \leq \mathbb{F}^3$ be a vector plane and $x \in \mathbb{F}^3$ be a vector not in H_0. Let $H_1 = x + H_0$, an affine translate of H_0. Clearly, $\mathbb{A}^2(\mathbb{F})$, H_0, and H_1 are isomorphic affine planes; in particular $\widehat{\mathbb{A}^2(\mathbb{F})}$ and $\widehat{H_1}$ are isomorphic projective planes. So, it suffices to see that $\widehat{H_1}$and $\mathbb{P}^2(\mathbb{F})$ are isomorphic, as follows.

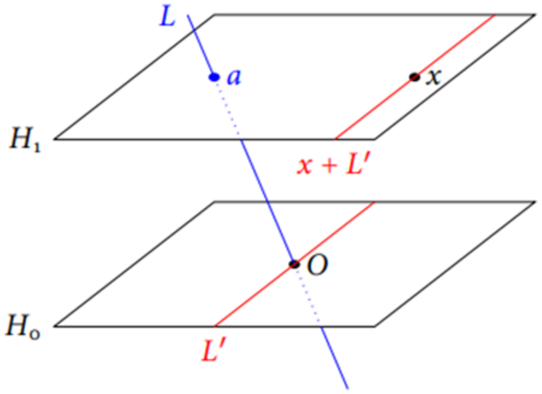

Figure 4.1. If $L \nleq H_0$ is mapped to a, while $L' \leq H_0$ is mapped to $[x + L']$

Let $L \leq \mathbb{F}^3$ be a vector line. If $L \nleq H_0$, then L will intersect H_1in a point say a; map L to a. If $L \leq H_0$, then there is no intersection, so L should be mapped to a point at infinity.
In the previous construction, such points were the equivalence classes (directions) of lines of H_1; so, map L to$[x + L]$, the direction of $x + L$ which is a line of the affine plane H_1. These maps are vector lines in F^3 to points in $\widehat{H}_1$. Now let $H \leq F^3$ be a vector plane. If $H \neq H_0$ then

$H \cap H_1$ is a line ℓ of the affine plane H_1 ; map H to $\hat{\ell}$ as in the projectivization construction. If on the other hand,$H = H_0$ then map H to ℓ_∞.
These maps vector planes in F^3 to lines in $\widehat{H}_1$.
The proof finishes by the following steps:

- Check that we have a bijection between points of $\mathbb{P}^2(\mathbb{F})$ and points of $\widehat{H_1}$;
- Check that we have a bijection between lines of $\mathbb{P}^2(\mathbb{F})$and lines of$\widehat{H_1}$;
- Check that these bijections preserve incidence.

5. Conclusions

Throughout this paper, we have examined the basic properties of affine and projective geometries. Firstly, we present the basic properties of affine and projective planes including their completion with each other, respectively, then we took fields and made both affine and projective geometries. Is shown that there can be constructed an affine plane $\mathbb{A}^2(\mathbb{F})$ from the ternary ring in a natural way. We constructed a projective plane given an affine plane and made an affine plane given a projective plane. Ordinary affine lines are not too short, so we must force a projective line to contain the direction. The converse operation of downgrading from projective to affine involves choosing the line to be removed.

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