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ON THE DYNAMIC OF HENON MAP. HOW TO CONTROL IT!

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Abstract

Most of the dynamics displayed by highly complicated nonlinear systems also appear in simple nonlinear systems. The purpose of the two-dimensional map with a strange attractor was for it to be a simple mapping that possesses similar properties to the Lorenz system and its Poincare map. Henon map is investigated, periodic points are found, and chaotic attractors are produced.
In this paper, we will demonstrate an orbit of the Henon map with 10000 points, which vary on the initial conditions of the orbit and the values of the two parameters of the system. Known that the chaotic attractors in the Henon map are neither area filling (dimension 2) nor a simple curve (of dimension 1), the dimensions of these complicated geometries must be non-integer values between 1 and 2, and the chaotic attractors are then called fractals or strange attractors. Thee capacity or box-counting dimension d_{box} is the simplest possible way to measure such pathologies. We use The OGY (Ott, Grebogi, and Yorke) Method with the idea to make small time-dependent linear perturbations to the control parameter *p* in order to nudge the state towards the stable manifold of the desired fixed point.
Acting on 500 equally spaced initial points (x_0,y_0) on a circle or a square, represent an numerical experiment which may give us some hints about why Jupiter's red spot and Saturn's hexagon-shaped hurricane seem to exist forever without contracting.

Keywords: Discrete nonlinear systems, Henon map, chaotic attractor, box-counting dimension, fixed point, OGY method.

1. Introduction

Two-dimensional iterated system given by

$$\begin{aligned} x_{n+1} &= 1 + y_n - \alpha x_n^2, \\ y_{n+1} &= \beta x_n, \qquad \alpha > 0, |\beta| < 1 \end{aligned} \tag{1.1}$$

named as Henon map, displays periodicity, mixing, and sensitivity to initial conditions [3]. The system can display hysteresis and bistability, which can be observed in the bifurcation diagrams [5]. The process starts from discrete nonlinear system $x_{n+1} = P(x_n, y_n)$, $y_{n+1} = Q(x_n, y_n)$ with fixed point (x_1, y_1), which can be transformed to the origin and the nonlinear terms can be discarded after taking a Taylor series expansion [1, 7]. The Jacobian matrix is given by:

$$J(x_1, y_1) = \begin{bmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{bmatrix}$$

(1.2)

Definition 1.1. For two eigenvalues of Jacobian matrix (1.2), λ_1, λ_2, a fixed point is called *hyperbolic* if both $|\lambda_1| \neq 1, |\lambda_2| \neq 1$. If either $|\lambda_1| = 1$ or $|\lambda_2| = 1$ then the fixed point is called nonhyperbolic [2, 3].

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The fixed point is stable as long as $|\lambda_1|<1$ and $|\lambda_2|<1$, otherwise the fixed point is unstable. For the Henon map can be found by solving the equations given by $x_{n+1}=x_n$ and $y_{n+1}=y_n$ simultaneously. Therefore, period-one points satisfy the equations $x=1-\alpha x^2+y,\ y=\beta x$, with solution $x=\frac{(\beta-1)\pm\sqrt{(1-\beta)^2+4\alpha}}{2\alpha}$, $y=\beta\left(\frac{(\beta-1)\pm\sqrt{(1-\beta)^2+4\alpha}}{2\alpha}\right)$. The map has two fixed points of period one if and only if $(1-\beta)^2+4\alpha>0$. Characteristic equation is $\lambda^2+2x\lambda-\beta=0$, then $\lambda=-x\pm\sqrt{x^2+\beta}$. When $|\lambda|>1\Rightarrow \alpha<\frac{-(1-\beta)^2}{4}$ or $\frac{3(\beta-1)^2}{4}<\alpha$ and $|\lambda|<1\Rightarrow\ \alpha>\frac{-(1-\beta)^2}{4}$ or $\frac{3(\beta-1)^2}{4}>\alpha$. If $\alpha>\frac{-(1-\beta)^2}{4}$ or $\frac{3(\beta-1)^2}{4}>\alpha$, then fixed points are sink, and, if $\alpha<\frac{-(1-\beta)^2}{4}$ or $\frac{3(\beta-1)^2}{4}<\alpha$, then fixed points are source [1,7].

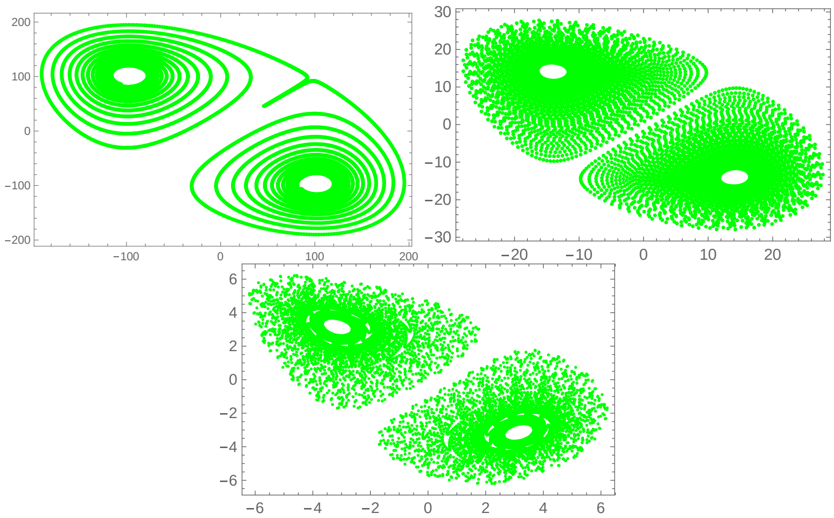
Theorem 1.1. [2] For $|\beta|<1$, such that $\alpha_0(\beta)=-\frac{1}{4}(\beta+1)^2$

a) $\alpha<\alpha_0(\beta)$, Henon map has no fixed point
b) $\alpha=\alpha_0(\beta)$, Henon map has exactly one fixed point
c) $\alpha>\alpha_0(\beta)$, Henon map has exactly two fixed points

The choice of initial conditions is important in these cases as some orbits are unbounded and move off to infinity. For the map, different chaotic attractors can exist simultaneously for a range of parameter values of α. This system also displays hysteresis for certain parameter values.

2. Orbit Diagram

For $\alpha\in(0,0.1]$ and $\beta\in[0.99,1)$ the map displays different behavior, the demonstration plots the first 20000 iterates of the Henon map from initial condition $(x_{0,}\ y_0)=(0,0)$.
For $\alpha=0.0001,\ \beta=0.99959$, $(1-\beta)^2+4\alpha=0.000400168>0$, the map (1.1) has two fixed points of one period, $A(97.971, 97.9308),\ B(-102.071,-102.029)$, with eigenvalues $(-1.00964, 0.99005)$ and $(1.01005,-0.98964)$ respectively, the points are hyperbolic. For $\alpha=0.005,\ \beta=0.99959$, $(1-\beta)^2+4\alpha=0.0200002>0$ the map (1.1) has two fixed point of one period , $A(14.1012, 14.0954),\ B(-14.1832,-14.1774)$, with eigenvalues $(-1.07278, 0.93177)$ and $(1.07322,-0.93139)$ respectively, it means that the points are hyperbolic.



a) b)

c)

Figure 2.1. Orbit Diagram of the Henon Map (1.1) a) $\alpha = 0.0001$ β=0.99959 , b) $\alpha = 0.005$ β=0.99959 c) $\alpha = 0.1$ β=0.99959

For different values of α such as $\alpha = 0.0001$, α=0.005, $\alpha = 0.1$ and fixed $\beta = 0.99959$, map (1.1) shows different forms of periodic behavior, for $\alpha = 0.0001$, Fig. 2.1. a), map (1.1) shows clean periodic behavior, for $\alpha = 0.005$, Fig. 2.1. b) the behavior is periodic with small tend to chaotic and for $\alpha = 0.1$, Fig. 2.1. c) the behavior of map (1.1) is chaotic periodic.
The standard demonstration of chaotic attractor of Henon map (1.1) is shown in Fig. 2.1, for $\beta = 0.4$ and allow the control parameter, in this case α, to vary around a nominal value greater than 1[6], (for $\alpha \in (0,1)$ doesn't exist an chaotic attractor) let say, $\alpha = 1.2$, for which the map has two fixed points of one period, $A(-4,,-2)$, $B(\frac{4}{3},\frac{2}{3})$, with eigenvalues $(-0.28,1.78)$ and $(-1,0.5)$ respectively, A is saddle point and B is nonhyperbolic, see Fig. 2.2.

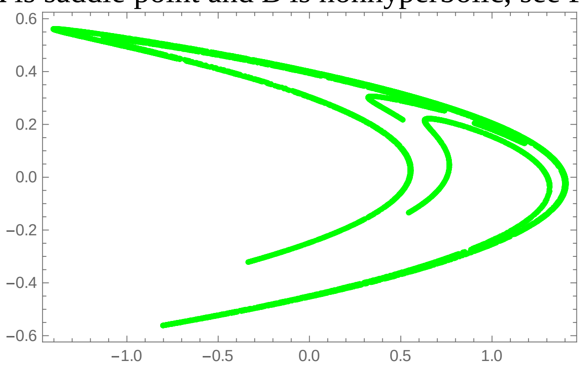


Figure 2.2. Iterative plot for the Hénon map (10000 iterations) when $\alpha = 1.2$ and $\beta = 0.4$ with initial conditions $(x_0, y_0) = (0,0)$

In this work we plot different demonstrations of Fig. 2.2., using box counting method to measure different pathologies of the map (1.1) [4], continuing with attempts to control the chaos using the OGY method [2]!
Using numerical experiment for fixed $\beta = 0.4$ and varied α there are fixed points with different number of periods, $\alpha = 0.2$ two fixed points of periods one, $\alpha = 0.5$ two fixed points

of period two and for $\alpha = 0.9$ two fixed points of period four! Using numerical experiment, we can say that the system exhibits chaotic behavior when $\alpha \in [1.16, 1.41]$ and $\beta = 0.4$.

2.1. Box counting diagram: In order to get a better understanding of the attractor, and of such shapes in general, we will numerically compute some it its fractal dimensions, which are dimensions that are not restricted to integers [8].

Definition 2.1. The dimension of $S \in \mathbb{R}^n$ (a bounded set) is a grid of *n*-dimensional boxes of side length ε over *S*. Let $N(\varepsilon)$ be equal to the number of boxes of the grid that intersect *S*. Then the scaling law for the dimension *d* give $d = \dfrac{\ln N(\varepsilon) - \ln C}{\ln 1/\varepsilon}$, where *C* is a constant for all small ε. The contribution of the second term in the numerator of this formula will be negligible for small ε [3, 4]. This demonstration shows the orbit diagram (OD) and the box counting diagram (BC) of the Henon map (1.1). The chaotic attractors are neither area filling (dimension 2) nor a simple curve (of dimension 1). Therefore, the dimensions of these complicated geometries must be non-integer values between 1 and 2, and the chaotic attractors are then called fractals or strange attractors. The capacity or box-counting dimension d_{box} is the simplest possible way to measure such pathologies [1,4]. It can be defined by:

$$d_{\text{box}} = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log\left(\frac{1}{\varepsilon}\right)} = \lim_{k \to \infty} \frac{\log N(k)}{\log\left(\frac{k}{\Delta}\right)} \tag{3.1}$$

where $N(\varepsilon) = N(k)$ is the number of boxes with size $\varepsilon = \varepsilon(k) = \dfrac{\Delta}{k}$ covering the attractor. Here $k \geq 1$ is the box-counting step and $\varepsilon(1) = \Delta$ is the size of the box at the initial step $k = 1$. Δ represents the sizes of the horizontal and vertical plot range, $\varepsilon(1) = x_{right} - x_{left} = y_{top} - y_{bottom} = \Delta$.

The method relies on covering the state space with a grid of boxes of side length ε and counting the number of boxes that contain elements of the respective set. We draw pictures and counting boxes, and use the results to form an estimate of the dimension [2, 8].

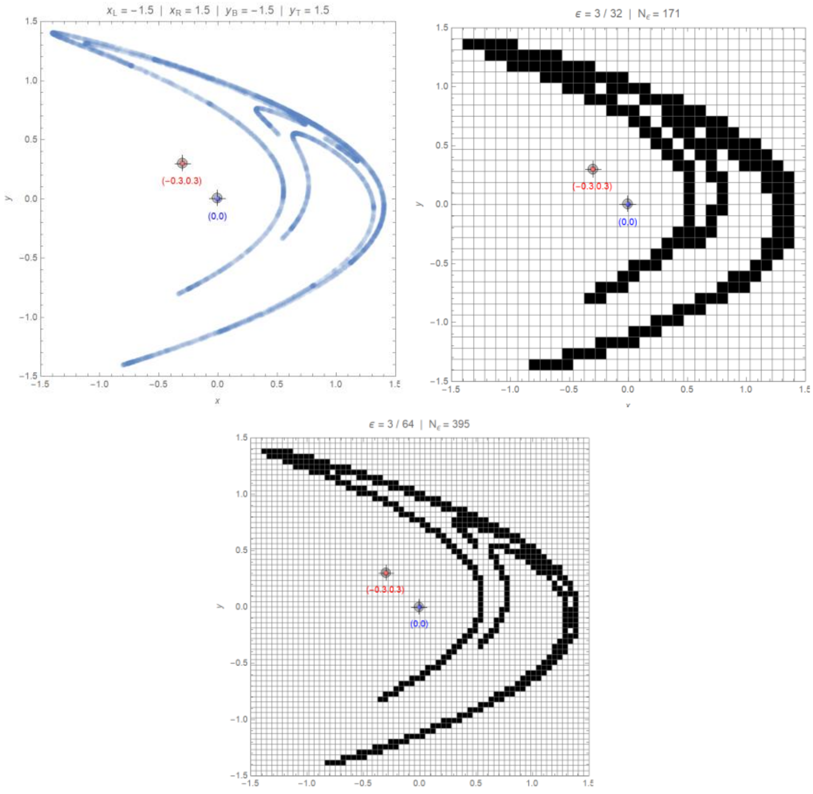
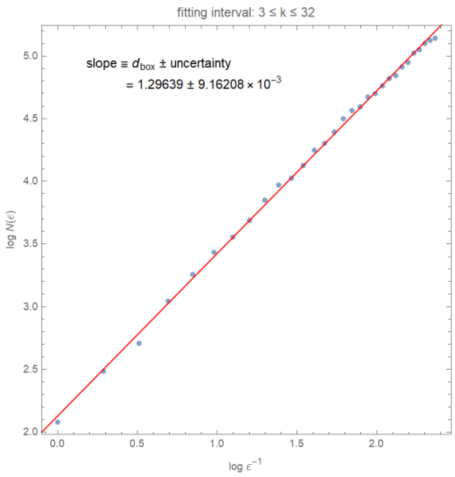


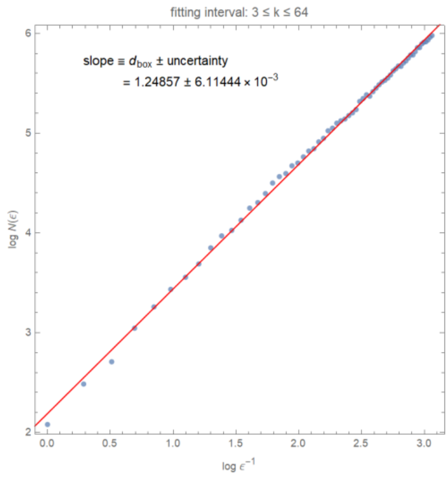
Figure 2.3. a) orbit diagram, b) box counting diagram with 1024 boxes, c) box counting diagram with 4096 boxes, of Henon map (1.1) for $\alpha = 1.2$ β=0.4, $\Delta = 3$, the number of iterations $n = 10000$, the number of initial iterations to be dropped $n_{\text{drop}} = 100$

Of the 1024 boxes shown, in Fig. 2.3. b), 168 contain a piece of attractor and of 4096 boxes shown, in Fig 2.3. c), 392 contain a piece of attractor. The side length of the boxes is $\varepsilon = \frac{3}{32}$, $\varepsilon = \frac{3}{64}$ respectively in the two plots.

Fig. 2.4. presents a plot of $\log N(\varepsilon)$ versus $\log \varepsilon^{-1}$, where the box-counting dimension is computed as the slope of the least-squares linear fit [8]. The box counting dimension would be extended as far as possible to the right, in order to make the best approximation possible to the limit. The slope in picture gives the value a) $1.29639 \pm 9.16208 \times 10^{-3}$, b) $1.24857 \pm 6.11444 \times 10^{-3}$



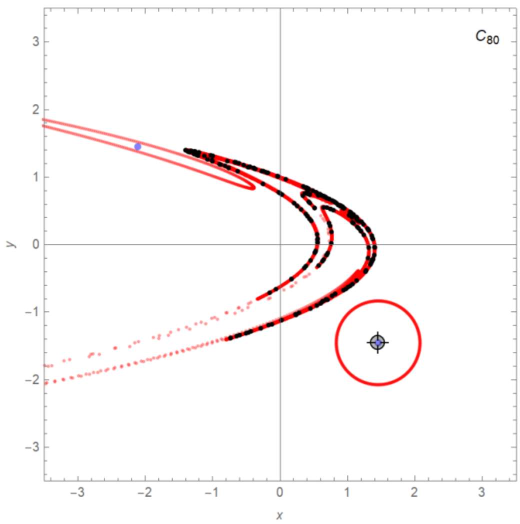
a)



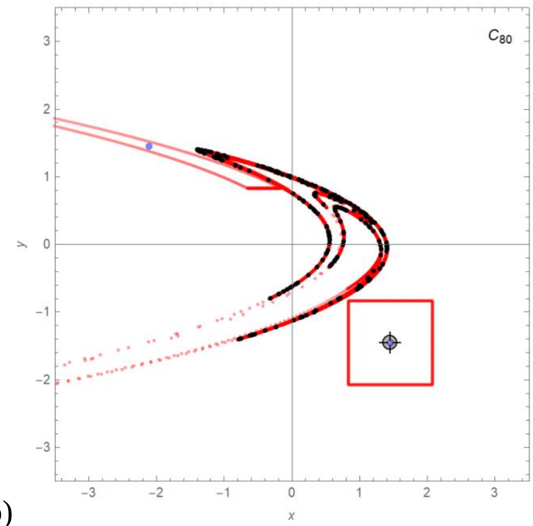
b)

Figure 2.4. Dimension estimation plot of Henon map (1.1) for $\alpha = 1.2$ β=0.4, $\Delta = 3$, the number of initial iterations to be dropped $n_{drop} = 100$, the number of iterations $n = 10000$.

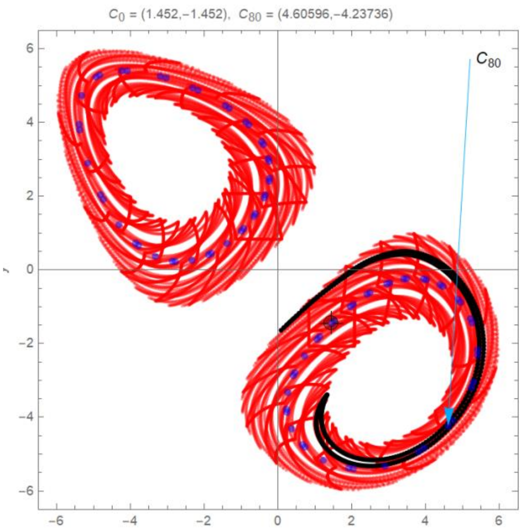
2.2.Henon Map Starting with a Circle or a Square: This Demonstration shows iterates of the dissipative Henon map (1.1) for $\beta = 0.4$ and $\alpha = 1.2$, acting on 500 equally spaced initial points (x_0, y_0) on a circle or a square. The shape and the color of these spirals become very close to those of the true yin-yang spiral by imagining that the initial circle is filled in with black dots. This numerical experiment may give us some hints about why Jupiter's red spot and Saturn's hexagon-shaped hurricane seem to exist forever without contracting or expanding.



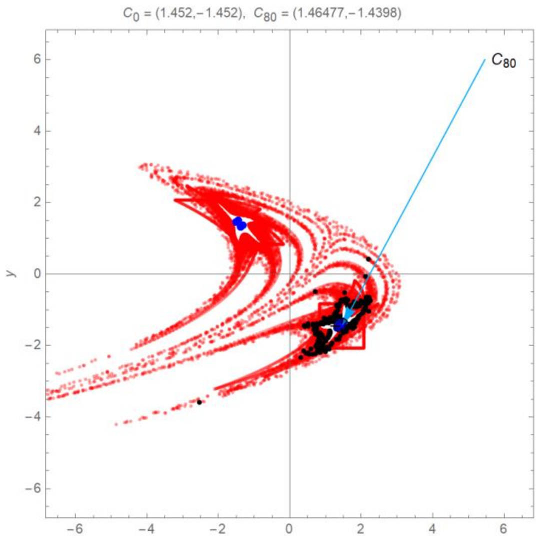
a)



b)



c)



d)

Figure 2.5. Iterates of the Henon map (1.1) acting on 500 equally spaced initial points $(x_0, y_0) = (0,0)$ on a circle or a square. a) for $\alpha = 1.2$ and $\beta = 0.4$ starting from a cyrcle, b) for $\alpha = 1.2$ and $\beta = 0.4$ starting from a square c)) for $\alpha = 0.1$ and $\beta = 0.99959$ starting from a square (see Fig. 1.1), d) for $\alpha = 0.5$ and $\beta = 0.99959$ starting from a square

3. Controlling Chaos in Henon map

Ott, Grebogi, and Yorke used the Henon map to illustrate the control method named OGY Method [2, 7]. The two-dimensional iterated map function is given by (1.1) using the transformations $x_n = \frac{1}{\alpha} X_n$, $y_n = \frac{\beta}{\alpha} Y_n$, it becomes:

$$X_{n+1} = \alpha + \beta Y_n - X_n^2,\; Y_{n+1} = X_n$$

(3.1)

The proof that (1.1) can be transformed into (3.1) is given in [?]. For $\beta = 0.4$ and $\alpha = 1.2$, the fixed points of period one are $A(0.8358, 0.8358)$ and $B(-1.4358, -1.4358)$. For values of α close to α_0 in a small neighborhood of *A*, (1.1) can be approximated by a linear map:

$$Z_{n+1} - Z_S(\alpha_0) = \mathrm{J}\left(Z_n - Z_S(\alpha_0)\right) + \mathrm{C}(\alpha - \alpha_0),\; Z_n = (X_n, Y_n)^T,\; A = Z_S(\alpha_0),\; \mathrm{C}=\begin{pmatrix} \frac{\partial P}{\partial \alpha} \\ \frac{\partial Q}{\partial \alpha} \end{pmatrix}$$

(3.2)

Assume in a small neighborhood of *A*, we have:

$$\alpha - \alpha_0 = -\mathrm{K}\left(Z_n - Z_S(\alpha_0)\right),\; \mathrm{K}=(k_1, k_2)^T$$

(3.3)

Substitute (3.3) into (3.2) to obtain:

$$Z_{n+1} - Z_S(\alpha_0) = (\text{J-CK})\left(Z_n - Z_S(\alpha_0)\right)$$

(3.4)

The fixed point at $A = Z_S(\alpha_0)$ is stable if the matrix $\mathbf{J} - \mathbf{CK}$ has eigenvalues with modulus less than unity. In this particular case, for $\lambda_1 = \pm 1$, $\lambda_2 = k_2 - 0.4$ and $\lambda_2 = -2.67156 - k_1 \Rightarrow k_2 = -k_1 - 2.27156$ and $\lambda_2 = -(k_2 - 0.4)$ and $\lambda_2 = -0.67156 - k_1 \Rightarrow k_2 = k_1 + 1.07156$, respectively. The stable eigenvalues (regulator poles) lie within a triangular region. The perturbed Henon map becomes:

$$X_{n+1} = \left(-k_1\left(X_n - X_{1,1}\right) - k_2\left(Y_n - Y_{1,1}\right) + \alpha_0\right) + \beta Y_n - X_n^2,\ Y_{n+1} = X_n$$

(3.5)

Applying equations (3.1) and (3.5) without and with control, respectively, it is possible to plot time series data for these maps.

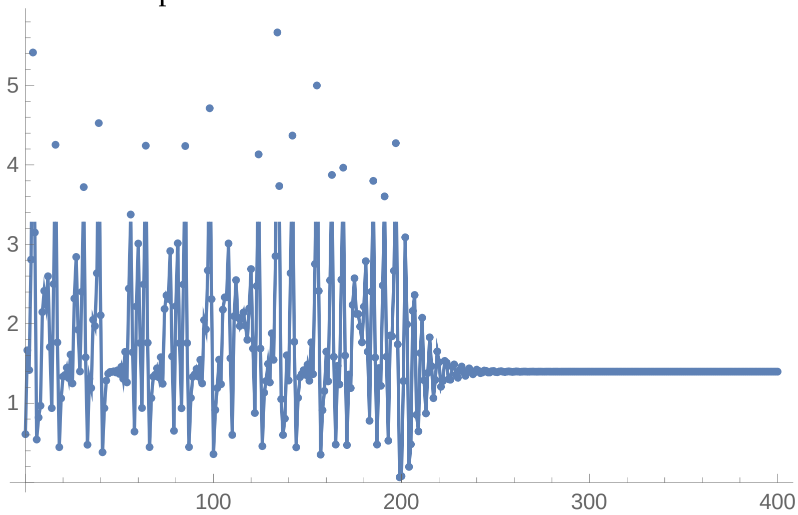


Figure 3.1. Time series data for the Henon map (1.1), for $\alpha = 1.2$ β=0.4, initial conditions $\left(x_0, y_0\right) = (0,0)$ with and without control, $r^2 = x^2 + y^2$. The control is activated after the 200th iterate.

4. Conclusions

In recent years, deterministic chaos has been observed when applying simple models to cardiology, chemical reactions, electronic circuits, laser technology, population dynamics, turbulence, and weather forecasting.

In this work we studied the dynamical behavior of the Henon map. We show the numerical values of the parameters where the map has an attractor and compare the different kinds of dimension of the maps. Using numerical experiment, we can say that the system exhibits chaotic behavior when $\alpha \in [1.16, 1.41]$ and $\beta = 0.4$. We use the box counting diagram to find the dimension of the map, although there is no single method of doing this. Lyapunov dimension and correlation dimension with common tools for the analysis of attractors from experimental data, are the next step to compare the measurement!

In the past, scientists have attempted to remove the chaos when applying the theory to physical models and for some systems, it has been found that the existence of chaotic behavior may even be desirable. The OGY technique is a feedback-control method. Some useful points to show about the algorithm are: the map can be constructed from experimental data. there may be more than one control parameter available and noise may affect the control algorithm. A simple blood cell population model represents a nonlinear dynamical system, an application of the map (1.1), with the red cell count per unit volume in the *n* the time interval and number of cells destroyed and produced in one-time interval, an option where our work may will be useful! We investigated the numerical results of the maps and used computer programming Mathematica for generating graphs and computations.

References

[1] M. S. Islam, "*Dynamical Behavior of Two Dimensional Henon Maps*", Bangladesh J. Sci. Ind. Res. 47(1), 55-60, 2012.

[2] S. Lynch, "*Dynamical Systems with Applications using Mathematica*", Birkhauser Boston, Springer Science, New York, 2007.

[3] M. Hénon, "A Two-Dimensional Mapping with a Strange Attractor," *Communications in Mathematical Physics*, **50**(1), 69–77, 1976.

[4] D. A. Russel, J. D. Hanson, and E. Ott, "Dimension of Strange Attractor," *Physical Review Letters*, **45**(14), 1175, 1980.

[5] H. Nagashima and Y. Baba, *Introduction to Chaos: Physics and Mathematics of Chaotic Phenomena*, Institute of Physics, Bristol, PA, 1999.

[6] T. Kapitaniak, *Controlling Chaos: Theoretical and Practical Methods in Non-Linear Dynamics*, Academic Press, New York, 1996.

[7] M. Benedicks and L. Carleson, "*The Dynamics of the Henon Map*", Ann. Math., (133), 73-169, 1991.

[8] T.J. Asbroek, "*The Henon map*", Bachelor’s Project, University of Groningen, 2023.