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Professional paper

APPLICATION OF CONIC SECTIONS IN PHYSICS AND ASTRONOMY

Rushadije HALILI[1], Endrita KELMENDI[2], Merita BAJRAMI[3]

[1]*Department of Mathematics, Faculty of Natural Sciences and Mathematics, NM*
[2] *Department of Mathematics, Faculty of Natural Sciences and Mathematics, NM*
[3] *Department of Mathematics, Faculty of Natural Sciences and Mathematics, NM*
rushadije.ramani@unite.edu.mk

Abstract

In this paper, we present the equations of some conic sections and explore their applications in physics and astronomy. We investigate ellipses, parabolas, circles and hyperbolas that are essential in modeling a range of phenomena, from planetary orbits and satellite trajectories to projectile motion and optical systems. By discussing these geometric forms and their relationships with key concepts in physics, we examine how conic sections contribute to our understanding of celestial mechanics, gravitational forces and light reflection. This paper aims to provide a foundation for further research on the significance of conic sections in scientific exploration.

Keywords: Conic sections, physic applications, astronomy applications.

1. Introduction

In this document, we'll take an intuitive approach to introduce and explain the classic definitions of conic sections. We'll then show how these definitions relate to the fields of physics and astronomy, making the concepts accessible and relevant.
Conic sections have a very rich history that goes all the way back to ancient Greece. Parabolas, ellipses, hyperbolas, and circles all have their origins there. Conic sections have been applied to our modern world in devising new technologies and investigating the universe. Many advancements in technology and science have come from the application of conic sections in physics and astronomy.

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Figure 14. Applications of Conic Sections in Physics and Astronomy

A cone is a three-dimensional geometric shape that tapers smoothly from a flat, circular base to a single point called the vertex. Basically, there are two types of cones *right circular cone* and *oblique cone.*

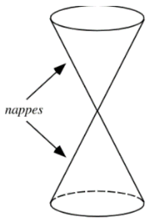

Figure 15.Cone

Conic sections have been studied since the time of the ancient Greeks and were thought to be an important mathematical concept. In this paper, we shall be discussing the four basic conic sections, some properties, their equations, and their application. The eccentricity of a conic section is defined to be the distance from any point on the conic section to its focus, divided by the perpendicular distance from that point in the nearest direction. This value is constant for any conic section and can also define the conic section:

- If $e = 0$, the conic is a circle
- If $e = 1$, the conic is a parabola.
- If $e < 1$, it is an ellipse.
- If $e > 1$, it is a hyperbola.

The directrix of a conic section is a line perpendicular to the axis that defines a conic section along with the focus. The distance of the point location from the focus is proportional to its horizontal distance from the directrix and is the constant of proportionality.

Figure 16.ellipse

Ellipse is the set of points in a plane whose distances from two fixed points, called foci. The foci of the ellipse are the two reference points $\boldsymbol{F_1 \ and \ F_2}$ that help draw the ellipse. If points $\boldsymbol{F_1 \ and \ F_2}$that are foci (multiple foci) and d is a given positive constant then $(\boldsymbol{x}, \boldsymbol{y})$ is a point on the ellipse if $\boldsymbol{d} = \boldsymbol{d_1} + \boldsymbol{d_2}$ as shown below:

The standard form of the equation of an ellipse centered at (0,0) follows:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

There are two such equations, one corresponding to the main horizontal axis and the other to the main vertical axis. The standard form of the equation of an ellipse with vertex at (h,k) follows:

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

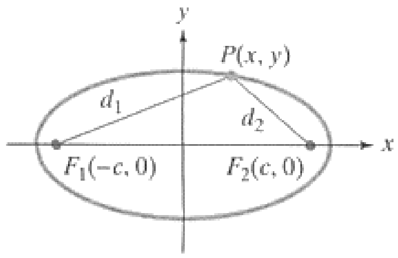

Figure 4.foci of ellipse

Figure 5. CIRCLE

A circle is the set of points in a plane that are equidistant from a given point, called the center.

$r^2=x^2+y^2$ or $x^2+y^2=r^2$

The equation of the circle in general form follows:

$$x^2+y^2+cx+dy+e=0$$

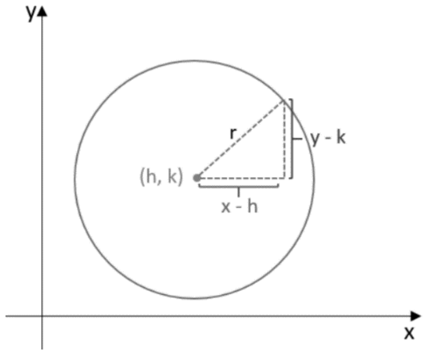

Figure 6.Circle

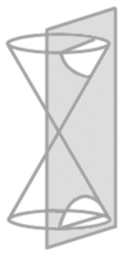

Figure 7. HYPERBOLA

A hyperbola is the set of points in a plane whose distances from two fixed points, called foci, have an absolute difference that is equal to a positive constant.

The standard form of the equation of the hyperbola with center at the origin and intercepts x (-a,0) and (a,0) follows,

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

where $a>0$ and $b>0$.The standard form of the equation of the hyperbola with center at the origin and intercepts y (0,-b) and (0,b) follows,

$$\frac{x^2}{b^2}-\frac{y^2}{a^2}=1\ where\ a>0\ and\ b>0$$

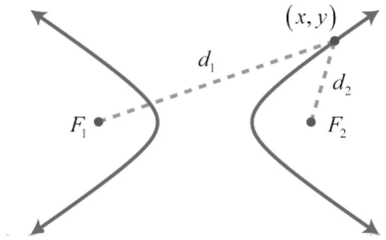

Figure 8. HYPERBOLA

Standard forms of equations of hyperbolas centered at (h, k):

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

$$(h \pm a, k)$$

and,

$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$$

$$(h, k \pm b)$$


Figure 9. PARABOLA

A parabola is the set of all points whose distance from a fixed point, called the focus, is equal to the distance from a fixed line, called the directrix. The point halfway between the focus and the directrix is called the vertex of the parabola.

The latus rectum is the chord of the parabola that is parallel to the directrix and passes through the focus

The standard form of the equation of a parabola with vertex at (h,k):

Vertical parabola: $y = a(x-h)^2 + k$

Horizontal parabola: $x = a(y-k)^2 + h$

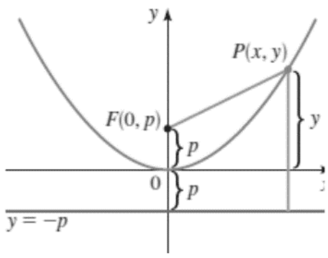

Figure 10..Parabola

2. Some Application of conic section in physics and astronomy

Conic sections play a crucial role in our understanding of physics and astronomy. They help us describe and predict many natural phenomena and the mechanics of celestial bodies. For instance, in physics, the paths of projectiles and particles moving under central forces can be represented by these shapes. In astronomy, conic sections are used to chart the movements of planets, comets, and other celestial objects, providing a framework for understanding how they travel through space. These shapes form the backbone of many concepts that explain the behavior of objects both on Earth and in the universe.

2.1 Elliptical orbits: Elliptical orbits are paths followed by objects as they move around a central point, typically a star or a planet. These orbits are not perfect circles but rather ellipses. Their shape is characterized by eccentricity (e), which measures how close or far the orbit is from being a perfect circle. In elliptical orbits, the eccentricity value falls between 0 and 1.

– When eccentricity is 0, the orbit is a perfect circle.
– As eccentricity approaches 1, the orbit becomes more elongated and less circular.

Within elliptical orbits, there are two key points:

– Apoapsis: the farthest point in the orbit from the center.
– Periapsis: the closest point in the orbit from the center.

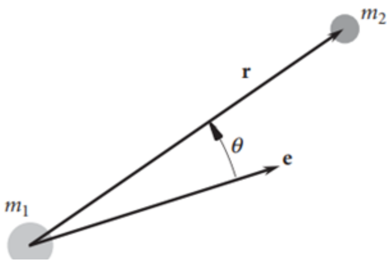

Figure 11. PATH

For orbits around a star, these points are referred to as aphelion and perihelion, respectively. In the case of planets orbiting the sun, these points play crucial roles in determining their orbital cycles.

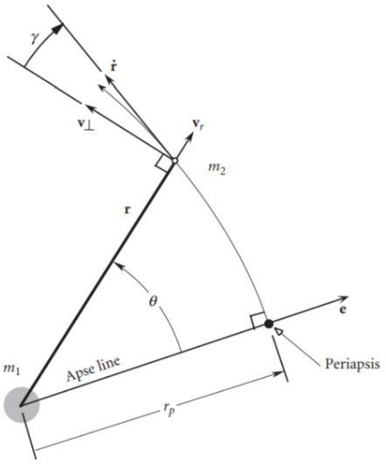

Figure 12. PATH

Elliptical orbits are commonplace in solar systems and constitute a significant aspect of Kepler's theory of planetary motion. This is the equation of the orbit and determines the path of body m_2 around m_1 , relative to m_1. Where μ represents the standard gravitational parameter, h is the specific relative angular momentum of the orbit, and e is a constant.

$$r + re\ cos\theta = \frac{h^2}{\mu},$$

$$r = \frac{h^2}{\mu} \frac{1}{1 + e\ cos\theta}$$

We see from this equation that m_2 approaches m_1 (r is smallest) when $\theta = 0$ (except when $e = 0$, in which case the distance between m_1 and m_2 is constant). The closest approach extends along the abscissa and is called periapsis. The distance rp to periapsis, as shown in the figure, gives us the equation:

$$r_p = \frac{h^2}{\mu} \frac{1}{1 + e}$$

It is clear that $vr = 0$ at periapsis. The flight path angle γ is also illustrated in the figure. This angle represents the angle that the velocity vector $v = r\grave{}$ makes with the normal vector.

$$tan\gamma = \frac{v_r}{v_\perp}$$

$$tan\gamma = \frac{e\ sin\theta}{1 + e\ cos\theta}$$

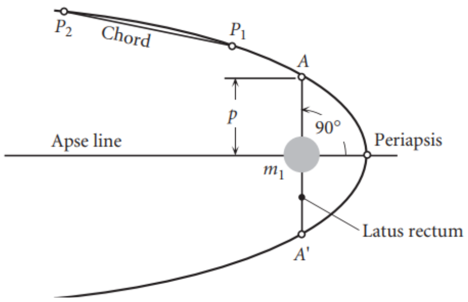

Figure 13. TRAJECTORY

Since $cos(-\theta) = cos\ \theta$, the trajectory described by the orbit equation is symmetric about the abscissa, as illustrated in the figure, which also shows a chord connecting any two points on the orbit. From the figure, we also observe the Latus rectum and semi-latus rectum. Otherwise, p is called the orbit parameter.

$$p = \frac{h^2}{\mu}$$

The maximum value of r is reached when the numerator takes its minimum value, which occurs at $\theta = 180°$. The maximum value of r is reached when the numerator takes its minimum value, which occurs at $\theta = 180°$. That point is called apoapsis, and its radial coordinate is denoted as r_a.

$$r_a = \frac{h^2}{\mu}\frac{1}{1-e}$$

Let $2a$ be the distance measured along the abscissa from periapsis P to apoapsis A, as illustrated in the figure.

From here we have:

$$2a = rp + ra$$

$$a = \frac{h^2}{\mu}\frac{1}{1-e^2}$$

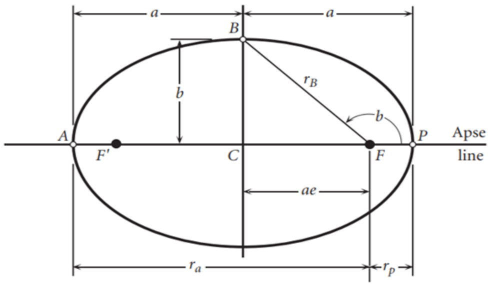

Figure 13. DISTANCE from periapsis P to apoapsis A

a is the major semi-axis of the ellipse. The solution of the equation for $\frac{h^2}{\mu}$ gives us:

$$r = a\frac{1-e^2}{1+ecos\theta}$$

Point F from the figure represents the location of body m_1, which is the origin of the r, θ polar coordinate system. The center C of the ellipse is the point extending along the middle of the path between apoapsis and periapsis. The distance CF from C to F is:

$$CF = a - FP = a - r_p$$

But knowing that:$r_p = a(1-e)$ then $CF = ae$

$$r_B = a\frac{1-e^2}{1+e\cos\beta}$$

The projection of r_B on the line of the apse is ae, that is:

$$ae = r_B\cos(180-\beta) = -r_B cos\beta = -a\frac{1-e^2}{1+e\cos\beta}cos\beta$$

Solving the equation for $e = -cos\beta$ whence we have:

$$r_B = a$$

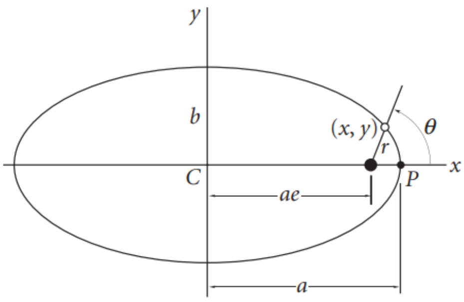

Figure 15.POINT IN ORBIT

Applying the Pythagorean Theorem:

$$b^2 = {r_B}^2 - (ae)^2 = a^2 - a^2e^2$$

At the point C we center a Cartesian xy coordinate system. In terms of r and θ, we see that the x coordinate of a point in the orbit is:

$$x = ae + rcos\theta = ae + (a\frac{1-e^2}{1+e\cos\theta})cos\theta = a\frac{e+cos\theta}{1+e\cos\theta}$$

From here we have: $\frac{x}{a} = \frac{e+cos\theta}{1+e\cos\theta}$

For the y coordinate we have:

$$y = rsin\theta = \left(a\frac{1-e^2}{1+e\cos\theta}\right)sin\theta = b\frac{\sqrt{1-e^2}}{1+e\ cos\theta}sin\theta$$

$$\text{Thus, } \frac{y}{b} = \frac{\sqrt{1-e^2}}{1+e\ cos\theta}sin\theta$$

Using these functions for the x and y coordinates we have:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = \frac{1}{(1+ecos\theta)^2}[(e+cos\theta)^2+(1-e^2)sin^2\theta]$$
$$= \frac{1}{(1+ecos\theta)^2}[e^2+2ecos\theta+cos^2\theta+sin^2\theta-e^2sin^2\theta]$$
$$= \frac{1}{(1+ecos\theta)^2}[e^2+2ecos\theta+1-e^2sin^2\theta]$$
$$= \frac{1}{(1+ecos\theta)^2}[e^2(1-sin^2\theta)+2ecos\theta+1]$$
$$= \frac{1}{(1+ecos\theta)^2}[e^2cos^2\theta+2ecos\theta+1] = \frac{1}{(1+ecos\theta)^2}(1+ecos\theta)^2$$

From where:

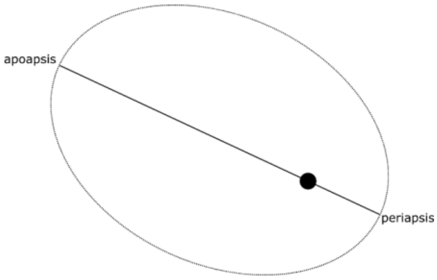

Figure 16.POINT IN ORBIT

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$$

This is the well-known Cartesian coordinate formula for an ellipse centered at the origin, with x intercept at $\pm a$ and y intercept at $\pm b$.

The speed varies depending on the point of the orbit (fastest at periapsis and slowest at apoapsis).

2.2 Parabolic orbits :

If the eccentricity is equal to 1, then the orbit equation takes the form:

$$r = \frac{h^2}{\mu}\frac{1}{1+cos\theta}.$$

We recall that the parameter p of an orbit is given by the equation:

$$p = \frac{h^2}{\mu}$$

Let's substitute that expression into the equation:

$$r = \frac{h^2}{\mu}\frac{1}{1+cos\theta}$$

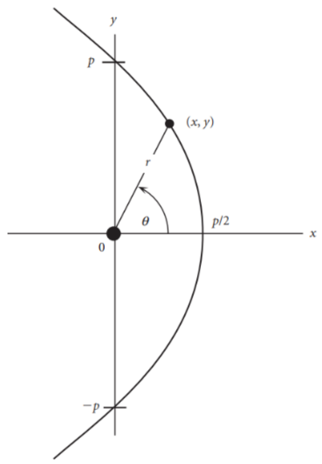

Figure 17. POINT IN PARABOILC ORBIT

and we present it $r = \frac{2a}{(1+cos\,\theta)}$ in a Cartesian system focused on the focus, as illustrated in the figure. Thus,

$$\frac{x}{\frac{p}{2}} + \frac{y^2}{p} = 2\frac{cos\theta}{1+cos\theta} + \frac{sin^2\theta}{(1+cos\theta)^2}$$

$$\frac{x}{\frac{p}{2}} + \frac{y^2}{p} = \frac{2cos\theta(1+cos\theta)+sin^2\theta}{(1+cos\theta)^2}$$

$$= \frac{2cos\theta + 2cos^2\theta + (1-cos^2\theta)}{(1+cos\theta)^2}$$

$$= \frac{1+2cos\theta+cos^2\theta}{(1+cos\theta)^2} = \frac{(1+cos\theta)^2}{(1+cos\theta)^2}$$

$$= 1$$

From where:

$$x = \frac{p}{2} - \frac{y}{2p}^2,$$

which is the equation of a parabola in a Cartesian coordinate system, the origin of which serves as the focus.

Example:

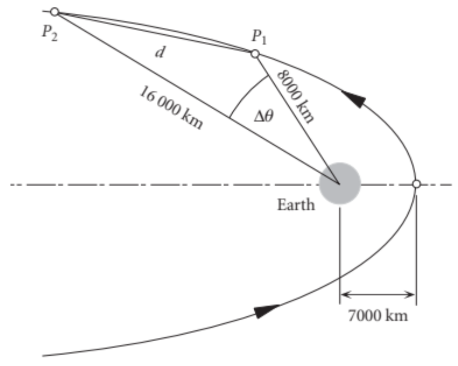

Figure 18. PARABOLIC geocentric trajectory

The perigee of a satellite in a parabolic geocentric trajectory is 7000 km. Find the distance d between points P1 and P2 on the orbit which are 8000 km and 16 000 km, respectively, from the center of the earth.

First, we calculate the angular momentum of the satellite by evaluating the orbit equation at periapsis,

$$r_p = \frac{h^2}{\mu}\frac{1}{1+cos\theta} = \frac{h^2}{2\mu}$$

from which

$$h = \sqrt{2\mu r_p} = \sqrt{2\cdot 398600\cdot 700} = 74700\ km^2/s$$

To find the length of the chord $\overline{P_1P_2}$, we must use the law of cosines from trigonometry,

$$d^2 = 8000^2 + 16000^2 - 2 \cdot 8000 \cdot 16000 cos\Delta\theta$$

The true anomalies of points P_1 and P_2 are found using the orbit equation:

$$8000 = \frac{74700^2}{398600}\frac{1}{1 + cos\theta_1} \rightarrow cos\theta_1 = 0.75 \rightarrow \theta_1 = 41.41°$$

$$16000 = \frac{74700^2}{398600}\frac{1}{1 + cos\theta_2} \rightarrow cos\theta_2 = -0.125 \rightarrow \theta_2 = 97.18°$$

Therefore, $\Delta\theta = 97.18° - 41.41° = 55.78°, so\ that\ d = 13270km$

2.3 Hyperbolic trajectories in celestial mechanics: In celestial mechanics, hyperbolic trajectories describe the paths of objects that pass near a massive body (such as a planet or star) and continue on an escape trajectory rather than being captured into an orbit. These trajectories are typically associated with objects such as comets, asteroids, or spacecraft performing gravity assists.

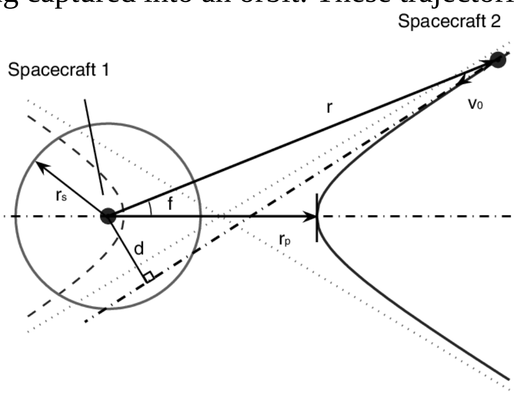

Figure 19. HYPERBOLIC trajectory

From the figure we see that Spacecraft 1 follows a hyperbolic trajectory towards the central body, influenced by its gravitational field. The initial trajectory is represented by the dashed lines approaching the central body.

As the spacecraft reaches the periapsis (r_p), it is at its closest point to the central body. The gravitational pull of the central body significantly affects the spacecraft's path and velocity. After passing the periapsis, the spacecraft follows a new hyperbolic trajectory, indicated by the outgoing dashed lines. The spacecraft enters the central body's sphere of influence on a hyperbolic trajectory.

The initial velocity and distance can be described by the hyperbolic trajectory equation:

$$r = \frac{1}{1 + ecos\theta}$$

$$v = \sqrt{2(\epsilon + \frac{\mu}{r})}$$

The closest approach to the central body, determined by:

$$r_p = \frac{1}{1 + e}$$

After the gravity assist, the spacecraft exits on a new hyperbolic trajectory with a new velocity. The new velocity can be described by the same energy equation, considering the velocity gained from the gravity assist:

$$v = \sqrt{2(\epsilon + \frac{\mu}{r})}$$

The hyperbolic excess velocity (v_∞)can also be calculated for the new trajectory:

$$v_\infty = \sqrt{\frac{2\epsilon}{m}}$$

- r is the distance from the focus (central body) to the spacecraft.
- e is the eccentricity of the hyperbola (for hyperbolas, e > 1).
- θ is the true anomaly (angle from the closest approach).
- ϵ specific orbital energy

3. Conclusion

The uses that conics have in physics and astronomy are very important to bridge abstractions in mathematics with actual occurrences. Thereafter, the study of their properties continues in the search for enrichment of theory and also leads to technology and exploration in these fields. The fact that they get this continued study and application suggests that further discoveries are in the offing. They provide a means of connecting theoretical abstractions in mathematics to real-life technological advances being made in the attempts to understand and explore this universe. As we continuously explore the depths of space and unravel the mysteries of the physical world, the principles of conic sections will stay with our journey and further increase the horizons for human knowledge

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