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CONNECTEDNESS WITH F-OPEN SETS

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Abstract

An *F*-open set is an open set with a finite boundary. Here it is defined the *F*-connectedness of a topological space as a space which cannot be expressed as a union of two non-empty disjoint *F*-open sets. It is shown that connectedness and *F*-connectedness are equivalent notions.

Keywords: Connectedness, F-open sets, Components, Quasicomponents.

1. Introduction

In [1], *F-open sets* of a topological space are presented as open sets with a finite boundary. In the same paper, *F-closed sets* are defined as closed sets with a finite boundary. By counterexamples it is shown that there exist open sets that are not *F*-open and closed sets which are not *F*-closed.
By using *F*-open sets, on the same paper, the notion of *F*-continuity and *F*-compactness are introduced. Here, we present and study the notion of *F*-connectedness. As a consequence, this improves a result that characterizes connectedness by using coverings and chains.

2. Preliminaries

Through the text, for a topological space X and $A \subseteq X$, by $\bar{A}$, $int(A)$ and ∂A we mean the closure, interior and boundary of *A*, respectively.
We present the definitions of *F*-open sets and *F*-closed sets as well as some of their properties and consequences stated in [1].

Definition 2.1: An open subset *A* of a topological space *X* is called *F-open* if $\bar{A} \setminus A$ is a finite set.

Definition 2.2: A closed subset *B* of a topological space *X* is called *F-closed* if $B \setminus int(B)$ is a finite set.

Since for an open set *A* and a closed set *B* we have that $A = int(A)$ and $B = \bar{B}$, then *A* is *F*-open if and only if ∂A is finite and *B* is *F*-closed if and only if ∂B is finite.
In [1], by counterexamples, it is shown that there exist open sets which are not *F*-open and closed sets which are not *F*-closed.
The question that may arise is if the family of *F*-open sets forms a topology in *X*. The answer is negative, since the countable union of *F*-open sets may not be *F*-open (see Example 6 in [1]).
It is also shown that a countable union of *F*-closed sets may not be *F*-closed and a countable intersection of *F*-open(*F*-closed) sets may not be *F*-open(*F*-closed). However, for a finite union

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or intersection of *F*-open or *F*-closed sets we have the following theorem:

Theorem 2.1: Let *X* be a topological space. Then:

- A finite union of *F*-open subsets is *F*-open;
- A finite union of *F*-closed subsets is *F*-closed;
- A finite intersection of *F*-open subsets is *F*-open;
- A finite intersection of *F*-closed subsets is *F*-closed.

Since the complement of an *F*-open set is *F*-closed and the complement of an *F*-closed set is *F*-open (see Theorem 1 in [1]), then:
Remark 2.1: A set is *F*-open and *F*-closed (*F*-clopen) in a topological space *X* if and only if it is clopen in *X*.

3. *F*-connectedness

By using *F*-open sets we can define *F*-connectedness of a topological space.
Definition 3.1: A topological space is *F-connected* if it cannot be expressed as a union of two nonempty disjointed *F*-open sets.
If there exist two nonempty disjointed *F*-open sets *A* and *B* such that $X = A \cup B$, then we say that *A* and *B* form an *F*-separation for *X*.
Since all *F*-open sets are open, connectedness of a topological space implies *F*-connectedness. However, if the space is *F*-connected, there may still exist two open sets (at least one of which is not *F*-open) which separate the space. Therefore, *F*-connectedness may not imply connectedness.
The following theorem, surprisingly, shows that *F*-connectedness and connectedness are equivalent.
Theorem 3.1: A topological space is connected if and only if it is *F*-connected.
Proof: The necessary condition is proved above.
Suppose that *X* is *F*-connected but there exist two nonempty disjoint open sets *A* and *B* such that at least one of them is not *F*-open. Without loss of generality, let *A* be an open set which is not *F*-open. Since *A* and *B* are a separation for *X*, they are both clopen, therefore $\partial A = \bar{A} \setminus A = \varnothing$ which contradicts *A* not being *F*-open. □
So, the topological space *X* is *F*-connected if and only if the only nonempty *F*-clopen (clopen) set in *X* is *X* itself.
Since continuous functions map connected spaces onto connected spaces and connectedness are inherited under homeomorphisms, by using Theorem 3.1 we have:
Corollary 3.1: If *X* is an *F*-connected topological space and $f : X \to Y$ a continuous function, then $f(X)$ is an *F*-connected subspace on *Y*. Moreover, if f is an onto function, then *Y* is *F*-connected.
Corollary 3.2: If $f : X \to Y$ is a homeomorphism, then *X* is *F*-connected if and only if *Y* is *F*-connected.
In [3] connectedness is characterized by using chains in open coverings of the topological spaces.
Theorem 3.2: A topological space *X* is connected if and only if for any two points *x* and *y* in *X* and for any open cover $\mathcal{U}$ of *X*, there exist $U_1, U_2, \ldots, U_n \in \mathcal{U}$ such that $x \in U_1$, $y \in U_n$ and $U_i \cap U_{i+1} \neq \varnothing$, for all $i \in \{1, 2, \ldots, n-1\}$.
In this case, the family $U_1, U_2, \ldots, U_n$ is called *a chain* in $\mathcal{U}$ joining *x* and *y*.
As a consequence, we have the following:

Theorem 3.3: A topological space *X* is *F*-connected if and only if for any two points *x* and *y* in *X* and for any cover $\mathcal{U}$ of *X* consisting of *F*-open sets, there exist $U_1, U_2, \ldots, U_n \in \mathcal{U}$ such that $x \in U_1$, $y \in U_n$ and $U_i \cap U_{i+1} \neq \varnothing$, for all $i \in \{1, 2, \ldots, n-1\}$.

Proof: Let *X* be an *F*-connected topological space such that there exists an open covering $\mathcal{U}$ of *X* consisting of *F*-open sets and two points $x, y \in X$ for which there is no chain in $\mathcal{U}$ joining them.

Let $A = \{z \in X \mid$ there is a chain in $\mathcal{U}$ joining *x* and $z\}$. Then *A* is open (even though it may not be *F*-open). *A* is also closed (even though it may not be *F*-closed) since for $z \in \bar{A}$, there exists $U_z \in \mathcal{U}$ such that $z \in U_z$ and $U_z \cap A \neq \varnothing$.

So, *A* is clopen, therefore $B = X \setminus A$ is also clopen. From Remark 2.1, *A* and *B* are *F*-clopen such that $x \in A, y \in B$, thus they form an *F*-separation for the topological space *X* which contradicts the *F*-connectedness of *X*.

Conversely, let *X* be a topological space such that for every open covering $\mathcal{U}$ of *X* consisting of *F*-open sets and any two points $x, y \in X$, there exists a chain in $\mathcal{U}$ joining them. Suppose that *X* is not *F*-connected, i.e., there exists an *F*-clopen set *A* and $x, y \in X$ such that $x \in A$ and $y \notin A$. Then $\{A, X \setminus A\}$ is a covering of *X* consisting of *F*-open sets for which there is no chain joining *x* and *y*, which contradicts the above statement. □

Theorem 3.3 improves the result in [3] by reducing the family of coverings of the space to only those coverings consisting of *F*-open sets.

Notice that every topological space *X* has at least one cover consisting of *F*-open sets, that is $\{X\}$. In cases when there is no other such cover (e.g., $\mathbb{R}^n, n > 1$ with the standard topology), Theorem 3.3 can be very useful.

4. F-components and F-quasicomponents

Components are the maximal connected sets of *X*. Similarly to components we can define *F*-components.

Definition 4.1: $A \subseteq X$ is said to be an *F-connected set* if it is *F*-connected as a subspace of *X*.

With other words, $A \subseteq X$ is *F*-connected if it cannot be expressed as a union of two nonempty disjoint *F*-open sets in the space *A* with the relative topology. As a consequence, from Theorem 3.1 we have:

Proposition 4.1: A set $A \subseteq X$ is *F*-connected if and only if it is connected.

Definition 4.2: The maximal *F*-connected set of *X* containing $x \in X$ is called the *F-component* of *x* in *X*.

As a direct consequence of Proposition 4.1 we have the following:

Theorem 4.1: Components and *F*-components coincide.

Since the image of a component under a continuous function lies on a component and homeomorphism induces a bijective mapping between components, as a consequence of Theorem 4.1 we have:

Corollary 4.1: If $f: X \to Y$ is continuous and C_X an *F*-component of *X*, then there exists C_Y an *F*-component of *Y* such that $f(C_X) \subseteq C_Y$.

Corollary 4.2: Let *CX* and *CY* be the sets of *F*-components of topological spaces *X* and *Y*, respectively. If $f: X \to Y$ is a homeomorphism, then $F: CX \to CY$ defined by $F(C) = f(C), \forall C \in CX$ is a bijection.

A *quasicomponent* of *x* in a topological space *X* is the intersection of all clopen sets containing *x*. Similarly, we define the *F*-quasicomponent of *x* in *X*.

Definition 4.3: The *F-quasicomponent* of *x* in a topological space *X* is the intersection of all *F*-clopen sets containing *x*.

Regarding Remark 2.1, we have the following consequence:

Theorem 4.2: *F*-quasicomponents and quasicomponents coincide.

Since any component is contained in a quasicomponent, by Theorems 4.1 and 4.2, we have:

Corollary 4.3: For every *F*-component *C* of a topological space *X* there is an *F*-quasicomponent *Q* of the same topological space such that $C \subseteq Q$.

Using the properties of quasicomponents and Theorem 4.2, similar to the Corollaries 4.1 and 4.2 we have:

Corollary 4.4: If $f: X \to Y$ is continuous and Q_X an *F*-quasicomponent of *X*, then there exists Q_Y an *F*-quasicomponent of *Y* such that $f(Q_X) \subseteq Q_Y$.

Corollary 4.5: Let *QX* and *QY* be the sets of *F*-quasicomponents of topological spaces *X* and *Y*, respectively. If $f: X \to Y$ is a homeomorphism, then $F: QX \to QY$ defined by $F(Q) = f(Q), \forall Q \in QX$ is a bijection.

Note that, like components and quasicomponenst, *F*-components and *F*-quasicomponenst form partitions of the topological space.

In [2-4] quasicomponents are characterized by using chains in open coverings of the topological spaces.

Theorem 4.3: Let *X* be a topological space and $x \in X$. The set $Ch(x) = \{y \in X \mid$ for every open cover U of *X*, there is a chain in U joining *x* and $y\}$ coincides with the quasicomponent $Q(x)$.

Next, we improve this theorem by taking just the family of coverings which consist only of *F*-open sets.

Theorem 4.4: Let *X* be a topological space and $x \in X$. The set $Ch(x) = \{y \in X \mid$ for every U an open cover of *X* consisting of *F*-open sets, there is a chain in U joining *x* and $y\}$ coincides with the quasicomponent $Q(x)$.

Proof: Let $\mathcal{U}$ be a cover of *X* consisting of *F*-open sets and let $Ch_{\mathcal{U}}(x) = \{y \in X \mid$ there is a chain in $\mathcal{U}$ joining *x* and $y\}$. As in the proof of Theorem 3.3, we can conclude that $Ch_{\mathcal{U}}(x)$ is clopen.

Next, we prove that if *A* is a clopen set in *X*, then there exist a cover $\mathcal{U}$ of *X* consisting of *F*-open sets such that $A = Ch_{\mathcal{U}}(x)$ for any $x \in A$. Since *A* is clopen, then *A* is *F*-clopen and $\mathcal{U} = \{A, X \setminus A\}$ is an open covering of *X* consisting of *F*-open sets such that $A = Ch_{\mathcal{U}}(x)$ for any $x \in A$.

So, *A* is clopen if and only if there is a covering $\mathcal{U}$ of *X* consisting of *F*-open sets such that $A = Ch_{\mathcal{U}}(x)$ for any $x \in A$.

Then, for $x \in X$, the set $\bigcap_{\mathcal{U}} Ch_{\mathcal{U}}(x) = \{y \in X \mid$ for every $\mathcal{U}$ an open cover of *X* consisting of *F*-open sets, there is a chain in $\mathcal{U}$ joining *x* and *y* $\} = Ch(x)$, where the intersection is over all coverings $\mathcal{U}$ consisting of *F*-open sets, is in fact the intersection of all clopen sets containing *x*, therefore $Ch(x) = Q(x)$. □

5. F-continuity and F-connectedness

In the previous sections we proved some results about preserving *F*-connectedness under continuous functions. In [1] the notion of *F*-continuity is presented.

Definition 5.1: A function $f : X \to Y$ is said to be *F-continuous* if $f^{-1}(U)$ is an *F*-open set in *X* for every open set *U* in *Y*.

It is shown that *F*-continuity implies continuity (Theorem 11 in [1]) but the converse may not hold (Example 17 in [1]).

Since *F*-continuity implies continuity, we have:

Corollary 5.1: If *X* is an *F*-connected space and $f : X \to Y$ is an *F*-continuous function, then $f(X)$ is an *F*-connected subspace of *Y*. Moreover, if f is an onto function, then *Y* is *F*-connected.

Corollary 5.2: If $f : X \to Y$ is an *F*-continuous function and C_X an *F*-component of *X*, then there exists C_Y an *F*-component of *Y* such that $f(C_X) \subseteq C_Y$.

Corollary 5.3: If $f : X \to Y$ is an *F*-continuous function and Q_X an *F*-quasicomponent of *X*, then there exists Q_Y an *F*-quasicomponent of *Y* such that $f(Q_X) \subseteq Q_Y$.

In the same paper the notion of *F*-homeomorphism is presented.

Definition 5.2: A bijective function $f : X \to Y$ is said to be an *F-homeomorphism* if f and f^{-1} are *F*-continuous.

Then, as a direct consequence of Corollary 5.1 we have:

Corollary 5.4: If $f: X \to Y$ is an *F*-homeomorphism, then *X* is *F*-connected if and only if *Y* is *F*-connected.

Corollary 5.5: Let CX and CY be the sets of *F*-components of topological spaces *X* and *Y*, respectively. If $f: X \to Y$ is an *F*-homeomorphism, then $F: CX \to CY$ defined by $F(C) = f(C), \forall C \in CX$ is a bijection.

Corollary 5.6: Let QX and QY be the sets of *F*-quasicomponents of topological spaces *X* and *Y*, respectively. If $f: X \to Y$ is an *F*-homeomorphism, then $F: QX \to QY$ defined by $F(Q) = f(Q), \forall Q \in QX$ is a bijection.

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