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QE IN RADICAL EQUATION AND SOLID GEOMETRY

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Abstract

The first elementary recursive real quantifier elimination procedure was based on cylindrical algebraic decomposition (CAD). A method of QE has a wide range of applications in many other areas. An interesting application in solving radical equations was presented through the examples. Also, an application in geometry, more precisely in the implicitization of parametric curves and surfaces was presented. A very complex example of implicitization of the Enneper surface was presented. This topic is significant in solid geometry; in problems of finding the intersection curve of two surfaces, it is the most convenient that one surface is presented by its implicit, while the other one is presented by parametric form.

1. Introduction

The first real quantifier elimination procedure was published by the author A. Tarski in [17]. During the 1970s G.E. Collins developed the first elementary recursive real quantifier elimination procedure [3,4] based on cylindrical algebraic decomposition (CAD). Its implementation was presented by D.S. Arnon [1]. After that period, CAD has undergone many improvements.

In this paper, we applied the QE method to solving radical equations and geometry problems. In the existing literature, related applications of real quantifier elimination methods include computational geometry and solid modelling [15,16], while there is no application of QE in solving radical equations.

A survey of the standard three implemented quantifier elimination methods was given in the paper [7] of authors A. Dolzmann, T. Sturm, and V. Weispfenning. A partial cylindrical algebraic decomposition was implemented in the program QEPCAD. The virtual substitution method was implemented in REDLOG. A complete elimination procedure based on Grobner bases in combination with multivariate real root counting, was implemented within the package QERRC. In practice, when testing their applicability to various problems in science, engineering, and economics, a conclusion was that none of the implementations is superior to the others. It can, in contrast, be necessary to combine all three of them to solve a problem. A good geometry example of combining all three mentioned QE methods is presented in the paper [6]. More precisely, a very difficult problem of an implicitization of the Enneper surface was solved.

Since it was shown that even trivial properties in plane geometry could have very difficult traditional proofs, applying the QE method to prove them could be very useful. This was a topic in the paper [18], where formulations of some geometry theorems were transformed into algebraic form and proved by QE. In this paper, we will present a different application of QE in geometry including in solid geometry. More precisely, a lot of examples of implicitization of parametric curves and surfaces were presented, including calculating the intersection point of two curves and the intersection curve of two surfaces.

Since the problems of determining the intersection curve of two surfaces are important in solid geometry, the application of the QE method can be very significant. In solving these tasks, it is

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the most convenient that one surface is presented by its implicit, while the other surface is presented by parametric form.
Considering the QE methods in this paper, a QE algorithm for a theory of RCF presented in [18] was used. An interesting application of this method of quantifier elimination in geometry and in solving radical equations was illustrated through the numerous examples. One complex problem, a problem of an implicitization of the Enneper surface was solved in this paper. It is formulated in [6]. A solution presented in [6] combines all three standard QE methods; QE by virtual substitution, Hermitian QE, and QE by partial cylindrical algebraic decomposition, which shows the difficulty of a problem.

2 Quantifier Elimination

Let us show the example of a formula with quantifiers which is equivalent to a formula without quantifiers.
Suppose we are given a formula φ (a,b,c) in a set of real numbers **R**,

$$\exists x(ax^2+bx+c=0).$$

By the quadratic formula, we have the following equivalence:

$$\varphi(a,b,c)\leftrightarrow[(a\neq 0 \wedge b^2-4ac\geq 0) \vee (a=0 \wedge (b\neq 0 \vee c=0))],$$

so φ is equivalent to a quantifier free formula.

Now let us introduce some basic definitions which are of importance for quantifier elimination. The language L is recursive if the set of codes for symbols from L is recursive. The first order theory T is recursive if the set of codes for axioms for T is recursive. An L-theory T is complete if for every sentence φ in a language L the following holds:

$$T\vdash \varphi \text{ or } T\vdash \neg\varphi.$$

For each theory T arises question of its decidability, i.e. the existence of algorithm which for given $\varphi \in Sent_L$ gives an answer whether $T\vdash \varphi$ or $T\nvdash \varphi$. In the case of recursive complete theory in a recursive language, the answer is affirmative.
A theory T of a language L admits quantifier elimination if for every formula $\varphi(\overline{v})\in For_L$ there exist a quantifier free formula $\psi(\overline{v})\in For_L$ such that:

$$T\vdash \forall v\left(\varphi(\overline{v})\leftrightarrow \psi(\overline{v})\right)$$

Every logic formula is equivalent to its following prenex normal form:

$$Q_1x_1...Q_nx_n\varphi(x_1,...,x_n,y_1,...,y_m),$$

where $Q_i \in\{\forall,\exists\}$ and φ is a formula without quantifiers in DNF; formula of the form $\forall x\varphi$ is equivalent to $\neg\exists x\neg\varphi$; $\exists x(\varphi\vee\psi)\leftrightarrow\exists x\varphi\vee\exists x\psi$ is a valid formula. Using the previous we see that an L-theory T admits quantifier elimination if and only if for every L-formula of the form $\exists x\varphi(\overline{y},x)$, where φ is a conjunction of atomic formulas and negations of atomic formulas, exists equivalent quantifier free formula $\psi(\overline{y})$.

The examples of theories which admit QE are the theory of dense linear order (DLO), theory of algebraically closed fields (ACF) and theory of real closed fields (RCF).

2.1 Theories of ACF and RCF: The language of fields is L = {+, -, ×, 0, 1, =}, where + and × are binary function symbols, - is unary function symbol, 0 and 1 are constant symbols and = is relational symbol.
We could axiomatize the class of algebraically closed fields by adding, to the axioms of fields (1), the axiom (2):

1. Axioms of field
2. for each n>1,

$$\forall x_0 \forall x_1 \cdots \forall x_{n-1} \exists x (x_0 + x_1 x + \cdots + x_{n-1} x^{n-1} + x^n = 0)$$

A set A= (1,2) is a set of axioms of algebraically closed fields; for any term *t* of a language L there exist a polynomial $p(x_1,...,x_n)$ with coefficients in **Z** such that $t = p(x_1,...,x_n)$ is a consequence of a set A. The set of axioms of algebraically closed fields allows quantifier elimination.
As an example of ACF, we can take the field of complex numbers, which is the algebraic closure of the field of real numbers.
In order to know how to eliminate quantifiers in a theory of algebraically closed fields, it is sufficient to know how to eliminate the existential quantifier in the formula of the form:

$$\exists x (t_1 = 0 \wedge \cdots \wedge t_k = 0 \wedge t \neq 0),$$

where t_i represent an atomic formula of a language L. So, every t_i is polinomial by *x* whose coefficients are polynomials by the other variables with coefficients in **Z**.
The language of ordered fields is L = {+, -,×, 0, 1, =, >}, where + and × are binary function symbols, - is unary function symbol, 0 and 1 are constant symbols and = and > are relational symbols.
We could axiomatize the class of real closed fields by adding, to the axioms of ordered fields (1), the axioms (2), (3):

1. Axioms of ordered field
2. $\forall x \exists y (x = y^2 \vee -x = y^2)$
3. $\forall x_0 \forall x_1 \cdots \forall x_{2n} \exists x (x_0 + x_1 x + \cdots + x_{2n} x^{2n} + x^{2n+1} = 0)$, for any n ≥1

Models of a set of axoms A = (1, 2,3) are real closed fields. The set A allows quantifier elimination; for any term *t* of a language L there exist a polynomial $p(x_1,...,x_n)$ with coefficients in **Z** such that $t = p(x_1,...,x_n)$ is a consequence of a set A.
The basic examples of a model of real closed fields are set of real numbers **R** and real closure of a set **Q**. The set A allows quantifier elimination.
In order to know how to eliminate quantifiers in a theory of real closed fields, it is sufficient to know how to eliminate the existential quantifier in the formula of the form:

$$\exists x \left(p_1 = 0 \wedge \cdots \wedge p_k = 0 \wedge q_1 > 0 \wedge \cdots \wedge q_m > 0 \right),$$

where p_i, q_j are polinomials by *x* whose coefficients are polynomials by the other variables with coefficients in **Z**.

3 Application of QE in solving Radical equations

While there are applications of QE in the existing literature in many other areas, that is not the case in solving radical equations. This fact gives strength to our work, as we introduce a new approach to solving radical equations.
We can apply a method of quantifier elimination to this problem successfully. Let us illustrate it through the examples.
Example 1 Solve the following radical equation:

$$\sqrt{x-16}+\frac{1}{2}\sqrt{x+16}=\frac{10}{\sqrt{x-16}}.$$

***Proof*:** Let us introduce the following notation:

$$y=\sqrt{x-16},\ z=\sqrt{x+16}$$

Now, the equation is equivalent to:

$$(\exists y)(\exists z)\left(y+\frac{1}{2}z=\frac{10}{y}\wedge y^2=x-16\wedge z^2=x+16\wedge y>0\wedge z\geq 0\right),$$

where it holds $x-16>0\wedge x+16\geq 0$. Let us evaluate z as a function of y from the equality $y+\frac{1}{2}z=\frac{10}{y}$. It follows:

$$z=\frac{20-2y^2}{y}$$

Now we substitute the previous value of z into the equality $z^2=x+16$. We have:

$$\frac{(20-2y^2)^2}{y^2}=x+16$$

After some basic calculation, the previous equality is equivalent to:

$$4y^4-96y^2-xy^2+400=0$$

Let us introduce the following notation:

$$t_1\equiv 4y^4-96y^2-xy^2+400,\ t_2\equiv\ y^2-x+16$$

We will apply the algorithm of QE to the following formula:

$$(\exists y)(t_1=0\wedge t_2=0\wedge y\neq 0)$$

Using the method for QE we have

$$T_1=A_2t_1-A_1y^2t_2,$$

where coefficients are equal $A_1=4,\ A_2=1$. Our formula is equivalent to:

$$A_2\neq 0\wedge(\exists y)(T_1=0\wedge t_2=0\wedge y\neq 0),\text{ or equivalently}$$

$$1\neq 0\wedge(\exists y)(3xy^2-160y^2+400=0\wedge y^2-x+16=0\wedge y\neq 0)$$

When we combine the two equalities from the previous formula we get a quadratic equation by a variable x. A resulting value of a variable x follows easily:

$$x = 20 \text{ or } x = 49\frac{1}{3}$$

Since we have a condition $x \leq 26$, the only solution is $x = 20$.

Example 2 Solve the following radical equation:

$$\sqrt[3]{2-x} = 1 - \sqrt{x-1}\,.$$

***Proof*:** Let us introduce the following notation:

$$y = \sqrt[3]{2-x},\ z = \sqrt{x-1}$$

Now, the equation is equivalent to:

$$(\exists y)(\exists z)(y^3 = 2 - x \wedge z^2 = x - 1 \wedge\ y = 1 - z \wedge x - 1 \geq 0 \wedge z \geq 0)$$

We can eliminate a variable y directly by substitution by a variable z. So, we have a formula that is equivalent to the previous one:

$$(\exists z)(-3z^2 + z(x+2) - x + 1 = 0 \wedge z^2 - x + 1 = 0\ \wedge x - 1 \geq 0 \wedge z \geq 0)$$

Let us introduce the following notation:

$$t_1 \equiv -3z^2 + z(x+2) - x + 1,\ t_2 \equiv\ z^2 - x + 1$$

After the application of QE algorithm by a variable z, we get a following equivalent formula:

$$(\exists z)(z(x+2) - 4x + 4 = 0 \wedge z^2 - x + 1 = 0\ \wedge x - 1 \geq 0 \wedge z \geq 0)$$

We can express a value of z from the first inner formula and substitute it into the second inner formula. After some basic calculation a result $x = 1$ follows easily.

We can also apply the same method to the radical equations with a parameter.

Example 3 Solve the following radical equation:

$$\frac{\sqrt{a+x} - \sqrt{a-x}}{\sqrt{a+x} + \sqrt{a-x}} = \frac{x}{a}\,, \qquad a \neq 0, x \neq 0, a > 0$$

***Proof*:** Let us introduce the following notation:

$$y = \sqrt{a+x},\ z = \sqrt{a-x}$$

Now, the equation is equivalent to:

$$(\exists y)(\exists z)\left(\frac{y-z}{y+z} = \frac{x}{a} \wedge y^2 = a + x \wedge z^2 = a - x \wedge a + x \geq 0 \wedge a - x \geq 0 \wedge y \geq 0 \wedge z \geq 0\right)$$

When we substitute a variable *x* by a variable *z* and apply the QE algorithm, we get only one result: $x = a$.

4 Application of QE in Geometry

There are a lot of geometry problems that cannot be solved using the quantifier elimination method. The main idea of this method is to eliminate quantifiers one by one. So, the problem is that the elimination of one quantifier can increase the degree of other quantified variables. Also, a large number of variables can represent a problem. To solve this problem, we combined a method for QE first presented in [18] and a substitution method in our paper.

4.1 Implicitization of Parametric curves and surfaces: It is determined that the implicit representation is the most convenient for determining if a given point belongs to a specific curve or surface. This motivates the search for finding the methods of converting from the parametric representation to the implicit one. The curves and surfaces which can be expressed implicitly in terms of polynomial equations are the algebraic curves and surfaces.
Now we will illustrate the implicitization of a planar curve presenting the example of a parametric quadric. Suppose we are given a general parametric form of the quadric:

$$x = a_2t^2 + a_1t + a_0$$
$$y = b_2t^2 + b_1t + b_0$$

We will find an implicit form by the method of quantifier elimination. So, we will apply QE to the following formula,

$$\exists t\left(a_2t^2 + a_1t + a_0 - x = 0 \wedge b_2t^2 + b_1t + b_0 - y = 0\right)$$

and get the formula equivalent to the previous one:

$$\exists t\left(t\left(a_1b_2 - a_2b_1\right) = b_2x - a_2y - a_0b_2 + a_2b_0 \wedge b_2t^2 + b_1t + b_0 - y = 0\right)$$

We can see that the first inner formula is linear by *t*. Let us introduce the following notation:

$$A = a_1b_2 - a_2b_1,$$
$$B = a_2b_0 - a_0b_2$$

It follows:

$$At = b_2x - a_2y + B,$$

or equivalently,

$$t = \frac{b_2x - a_2y + B}{A}$$

Now we substitute a value of *t* into the first coordinate equation:

$$x = a_2\left(\frac{b_2x - a_2y + B}{A}\right)^2 + a_1\frac{b_2x - a_2y + B}{A} + a_0$$

After some basic calculation we get the resulting implicit form:

$$a_2b_2^{\,2}x^2 + a_2^{\,3}y^2 - 2a_2^{\,2}b_2xy + \left(2a_2b_2B + a_1Ab_2 - A^2\right)x - \left(2a_2^{\,2}B + a_1Aa_2\right)y + a_1AB + a_0A^2 + a_2B^2 = 0.$$

Now we will illustrate the implicitization of a rational parametric curve presenting an example of a rational parametric quadric. Suppose we are given a general parametric form of a rational quadric:

$$x = \frac{a_2 t^2 + a_1 t + a_0}{d_2 t^2 + d_1 t + d_0}$$

$$y = \frac{b_2 t^2 + b_1 t + b_0}{d_2 t^2 + d_1 t + d_0}$$

We can write the previous equations in the following form:

$$(d_2 x - a_2)t^2 + (d_1 x - a_1)t + (d_0 x - a_0) = 0$$
$$(d_2 y - b_2)t^2 + (d_1 y - b_1)t + (d_0 y - a_0) = 0$$

Without loss of generality, we will solve this problem for concrete values of the coefficients:

$$x = \frac{2t^2 + 4t + 5}{t^2 + 2t + 3},$$
$$y = \frac{3t^2 + t + 4}{t^2 + 2t + 3}$$

or equivalently,

$$(x-2)t^2 + (2x-4)t + (3x-5) = 0$$
$$(y-3)t^2 + (2y-1)t + (3y-4) = 0$$

When applying QE method to the previous two formulas, we eliminate *t* and get a resulting implicit form:

$$50x^2 + y^2 - 175x - 6y + 159 = 0.$$

Now we will present two examples of implicitization of parametric surfaces. The examples are given below.

Example 4 Suppose we are given the following description of a torus,

$$\tau(x, y, z, r_1, r_2) \equiv \exists u\left(u^2 + z^2 = r_2^{\ 2} \wedge (r_1 + u)^2 = x^2 + y^2\right)$$

and we need to find its implicit form.

***Proof*:** We will apply quantifier elimination method in order to solve this example. So, we are given the following formula:

$$\exists u\left(u^2 + z^2 = r_2^{\ 2} \wedge (r_1 + u)^2 = x^2 + y^2\right).$$

When applying QE, it is equivalent to:

$$\exists u\left(x^2 + y^2 + z^2 - r_1^{\ 2} - r_2^{\ 2} - 2r_1 u = 0 \wedge (r_1 + u)^2 = x^2 + y^2\right) \wedge r_2^{\ 2} - z^2 \geq 0$$

We can see that the first inner formula is linear by *u*. So, we can express a value of *u*,

$$u = \frac{x^2 + y^2 + z^2 - r_1^{\,2} - r_2^{\,2}}{2r_1}$$

and substitute it into the equation $u^2 + z^2 = r_2^{\,2}$. After some basic calculation, we get an implicit form of a torus:

$$x^4 + y^4 + z^4 + 2x^2y^2 + 2x^2z^2 + 2y^2z^2 - 2x^2r_1^{\,2} - 2y^2r_1^{\,2} + 2z^2r_1^{\,2} +$$
$$+ 2z^2r_1^{\,2} - 2x^2r_2^{\,2} - 2y^2r_2^{\,2} - 2z^2r_2^{\,2} + +r_1^{\,4} + r_2^{\,4} - 2r_1^{\,2}r_2^{\,2} = 0.$$

Example 5 Suppose we are given a parametric form of the unit sphere,

$$x(s,t) = \frac{1 - s^2 - t^2}{1 + s^2 + t^2}$$
$$y(s,t) = \frac{2s}{1 + s^2 + t^2}$$
$$z(s,t) = \frac{2t}{1 + s^2 + t^2},$$

and we need to find its implicit form.

***Proof*:** We will apply quantifier elimination method in order to solve this example. Let us rewrite the coordinate formulas first.

$$t^2(x+1) + s^2x + x + s^2 - 1 = 0$$
$$t^2y + s^2y + y - 2s = 0$$
$$t^2z - 2t + s^2z + z = 0$$

Obviously, when we combine the second and the third equation we have the equality:

$$t = \frac{sz}{y}.$$

Now we will denote the first equation by $t_1 = 0$ and the second one by $t_2 = 0$. Our goal is to eliminate a parameter *t* from the following formula:

$$\exists t(t_1 = 0 \wedge t_2 = 0).$$

When we apply the algorithm for QE we get that the previous formula is equivalent to the following one:

$$\exists t(2sx + 2s - 2y = 0 \wedge t_2 = 0)$$

From the first inner formula of the previous one directly follows:

$$s(x+1) = y \Leftrightarrow s = \frac{y}{x+1}.$$

Since it holds: $t = \frac{sz}{y}$, we can substitute a value of *s* and get the following:

$$t = \frac{z}{x+1}.$$

Now we substitute values of *s* and *t* into the third equation (equivalent to a coordinate equation for a variable z). It follows:

$$\frac{z}{x+1}\, z\left(1+\left(\frac{y}{x+1}\right)^2+\left(\frac{z}{x+1}\right)^2\right)=2\frac{z}{x+1}.$$

After some basic algebraic calculation, the previous equation is equivalent to the following one:

$$x^2 + y^2 + z^2 = 1,$$

which represent an implicit form of the unit sphere.

We can point out that some curves do not have a parametric form; for example, cubic curves. Only singular cubic curves are parameterizable. Also, it is not possible to apply the QE method described in this paper to find the intersection point of a cubic curve in a general form and a line. If we would try to apply this method all terms would be canceled. The cubic curves are not significant in geometry, but they have a very important application in cryptography.
If we consider a reverse process, converting an equation of a surface from an implicit to a parametric form, the general problem is algorithmically unsolved.
The following example *6* is very difficult and represents an implicitization of the Enneper surface. A problem was formulated in the paper [6]. The author A. Dolzmann found an automatic solution of an implicitization of the Enneper surface by combining all three standard QE methods; QE by virtual substitution, Hermitian QE, and QE by partial cylindrical algebraic decomposition. The simplification methods were also used in [6] which shows the complexity of this problem.
A presented solution in this paper is original. A QE method presented in [18] was used. Also, it was necessary to combine the equations and use algebraic simplification in order to find a solution.

Example 6 Suppose we are given a parametric form of the Enneper surface:

$$\begin{aligned} x &= 3x_1 + 3x_1x_2^{\,2} - x_1^{\,3} \\ y &= -x_2^{\,3} + 3x_2 + 3x_1^{\,2}x_2 \\ z &= 3x_1^{\,2} - 3x_2^{\,2} \end{aligned}$$

and we need to find its implicit form.

***Proof*:** We will consider the second and the third equation and apply QE algorithm.

$$\begin{aligned} t_1 &\equiv -x_2^{\,3} + 3x_2 + 3x_1^{\,2}x_2 - y = 0 \\ t_2 &\equiv -3x_2^{\,2} + 3x_1^{\,2} - z = 0 \end{aligned}$$

A resulting inner formula is the following one:

$$T_1 \equiv -6x_1^{\ 2}x_2 - 9x_2 - zx_2 + 3y = 0$$

Now we apply QE on the first and the third given coordinate equation.

$$t_1 \equiv -x_1^{\ 3} + 3x_1x_2^{\ 2} - x = 0$$
$$t_2 \equiv -3x_2^{\ 2} + 3x_1^{\ 2} - z = 0$$

A resulting inner formula is the following one:

$$6x_1x_2^{\ 2} + 9x_1 - x_1z - 3x = 0$$

Now we combine the previous two resulting formulas. It follows:

$$3z - x_1^{\ 2}z - x_2^{\ 2}z - 3x_1x + 3x_2y = 0$$

When we express a variable x_2 from the third given equation (for z) and substitute it into the previous formula we get:

$$9z - 6x_1^{\ 2}z + z^2 - 9x_1x + 9x_2y = 0$$

Now we will consider the first two equations given in the example and apply QE algorithm.

$$t_1 \equiv -x_1^{\ 3} + 3x_1x_2^{\ 2} + 3x_1 - x = 0$$
$$t_2 \equiv -x_2^{\ 3} + 3x_2 + 3x_1^{\ 2}x_2 - y = 0$$

A resulting inner formula is the following one:

$$8x_1x_2^{\ 3} + 12x_1x_2 - 3x_2x - x_1y = 0$$

Now we combine the previously obtained equations and get the following formula:

$$4x_1x_2z + 3x_2x - 3x_1y = 0$$

If we substitute a value from the previous equation

$$x_1x_2 = \frac{3x_1y - 3x_2x}{4z}$$

into the following two equations (previously obtained),

$$6x_1x_2^{\ 2} + 9x_1 - x_1z - 3x = 0$$

$$-6x_1^{\ 2}x_2 - 9x_2 - zx_2 + 3y = 0$$

we get the following equivalent system:

$$9x_1x_2y - 9x_2{}^2x + 18x_1z - 2x_1z^2 - 6xz = 0$$

$$9x_1x_2x - 9x_1{}^2y - 18x_2z - 2x_2z^2 + 6yz = 0$$

Now we apply the same substitution for x_1x_2 into the previous system; so we can express the values of $x_2{}^2$ and x_1^2 from the equations of a system. When we substitute these values of x_1^2 and $x_2{}^2$ respectively, into a previously obtained formula,

$$3z - x_1{}^2z - x_2{}^2z - 3x_1x + 3x_2y = 0$$

we get a resulting system that is linear by variables x_1 and x_2:

$$-81xyx_1 + (8z^3+72z^2 + 27x^2 + 54y^2)x_2 = 18yz^2 - 54yz$$

$$(8z^3-72z^2 - 27y^2 - 54x^2)x_1 + 81xyx_2 = -18xz^2 - 54xz$$

Since it represents a system of two linear equation by the unknown variables x_1 and x_2, we can solve it easily and a resulting formula follows.

4.2 Intersection of two curves or two surfaces: Now we will present the application of QE method in finding the intersection point of two curves. It is presented in the following example *7*.

Example 7 Suppose we are given the following circle and elliptic line (ellipse):

$$l_1 : (x-p)^2 + (y-q)^2 = 1$$
$$l_2 : \frac{x^2}{4^2} + \frac{y^2}{2^2} = 1$$

If we are given that the two curves have only one intersection point, evaluate coordinates of that point.

***Proof*:** Let us rewrite a given equations first,

$$y^2 - 2qy + x^2 - 2px + p^2 + q^2 - 1 = 0$$
$$x^2 + 4y^2 - 16 = 0.$$

We will write a quantified formula in which we express the existence of the intersection point:

$$\exists x \exists y \left(y^2 - 2qy + x^2 - 2px + p^2 + q^2 - 1 = 0 \wedge x^2 + 4y^2 - 16 = 0\right)$$

If we apply the QE method to the inner formula we will decrease a power of a variable *y*. So, the previous formula is equivalent to:

$$\exists x \exists y \left(3x^2 - 8px - 8qy + 4p^2 + 4q^2 + 12 = 0 \wedge x^2 + 4y^2 - 16 = 0\right).$$

Now we apply QE method by a variable *x* to this inner formula of a previous one:

$$\exists x\left(3x^2 - 8px - 8qy + 4p^2 + 4q^2 + 12 = 0 \wedge x^2 + 4y^2 - 16 = 0\right).$$

It follows that the inner formula is equivalent to:

$$\exists x\left(3y^2 + 2qy + 2px - p^2 - q^2 - 15 = 0 \wedge x^2 + 4y^2 - 16 = 0\right).$$

The first subformula of a matrix formula represent a quadratic equation by a variable y. The quadratic equation has a unique solution if a discriminant D is equal zero. So, we will set up the equation:

$$D = (2q)^2 - 4 \cdot 3\left(2px - p^2 - q^2 - 15\right) = 0$$

After some basic calculation we get a value of a coordinate x: $x = \frac{3p^2 + 4q^2 + 45}{6p}$.

Similarly, we can evaluate a value of a coordinate y: $y = \frac{-p^2 + 3q^2 + 9}{6q}$

Note that we can also solve a problem when two given curves have more than one intersection point or have no intersection points.

QE method can be also applied in the problems of determining the intersection curve of two surfaces, which represents a very important problem in solid geometry. In solving of these tasks, it is the most convenient that one surface is presented by its implicit, while the other surface is presented by parametric form. We will illustrate this by the following example.

Example 8 Find the intersection of an upper half of a sphere $x^2 + y^2 + z^2 = 1$ and a cone $x^2 + y^2 = z^2$.

***Proof*:** We will use a parametric form of a cone in this example:

$$\begin{aligned} x &= rcos\varphi \\ y &= rsin\varphi \\ z &= r, \qquad 0 \le \varphi \le 2\pi \end{aligned}$$

When we substitute values for coordinates x, y, z into the equation of a sphere we get:

$$r^2cos^2\varphi + r^2sin^2\varphi + r^2 = 1$$

We can find a value of r from the previous equation:

$$r = \frac{1}{\sqrt{2}}$$

Obviously, a resulting intersection represents the following circle line:

$$x^2 + y^2 = \frac{1}{(\sqrt{2})^2}$$

with a centre $(0, 0, 1/\sqrt{2})$.

Similarly, we can find the intersection of a torus and a sphere. We can see in the ***Example 4*** how complex an implicit form of a torus is. So, we will use its parametric form. After some basic calculation we get a resulting formula for u:

$$u = \frac{R^2 - {r_1}^2 - {r_2}^2}{2(r_1 - r_2)}$$

Now we can substitute the previous value of u into the first inner formula of a torus and find a value of z. A resulting intersection formula follows easily.

4 Conclusion

Considering the application of quantifier elimination, some interesting examples of solving radical equations were presented in this paper. Also, the QE method could be very useful in finding an implicit form of the parametric curves and surfaces. A very complex example of calculating an implicit form of the Enneper surface was presented as a result. Its complexity can be seen in a different approach from the other authors [6]. In this paper, some examples of calculating the intersection point of two curves or the intersection curve of two surfaces were shown. The importance of finding the intersection curve in solid geometry motivated us to investigate the conversion of a parametric representation of a surface into its implicit form by QE methods presented in [18].

Unfortunately, there are a lot of geometry problems that cannot be solved using any quantifier elimination method. Even when we combine a method for QE and a substitution method, it is impossible to calculate a result for a large number of geometry problems. The main reason is that the elimination of one quantifier can increase the degree of other quantified variables. Since this topic is very significant, it has a lot of potential for future work.

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