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RÖSSLER SYSTEM AND ITS CHAOS, NUMERICALLY STUDIED USING FRACTIONAL ADAMS BASHFORTH MOULTON METHOD

Ylldrita SALIHI [1], Krutan RASIMI[2]

[1]*Department of Mathematics, Faculty of Natural Sciences and Mathematics, North Macedonia*
[2]*Department of Mathematics, Faculty of Natural Sciences and Mathematics, North Macedonia*

Abstract

The dynamics of the nonlinear fractional-order Rössler system are investigated through numerical simulations using the Fractional Adams–Bashforth–Moulton Method (FABM). To demonstrate the computational approach, the algorithm is applied to obtain a three-dimensional solution of the Rössler system, originally conceptualized by Otto Rössler as a mathematical model of a taffy-pulling machine. A fixed-parameter dynamical analysis, along with a chaos diagram, is conducted. The findings reveal that the fractional-order Rössler system exhibits complex and diverse dynamical behaviors, highlighting its potential for various applications. The fractional derivatives are described in the Caputo sense. Firstly, investigation of dynamics is realized by fixing the parameters $a=0.2,\ b=0.2,\ c=5.7$ (system has chaotic behavior), implemented with the aid of Mathematica symbolic package, system shows chaotic behavior for $v>0.839$. By varying with parameter $a=(0,\ 0.6)$ and fixed $b=2,\ c=4$, *v*=0.9 based on FABM, is shown that the system has rich dynamical characteristics with different types of chaos! For *a*=0.01 and *a*=0.56231, chaos changes from periodic cycles to random chaos (deterministic chaos) for the intervals $a=(0,\ 0.2)$ and $a=(0.2,\ 0.6)$ respectively and $v\in(0,1)$. An active control law is applied to the incommensurate fractional order Rössler system (*a*=0.5, *b*=2, *c*=4), using only one input. This indicates that the proposed controller can linearize the system and has stabilized it for the found value of $a=0.5\in(0.2,\ 0.6)$ where the system exhibits chaotic behavior.

Keywords: Caputo fractional derivative, Rössler system, Fractional Adams Bashforth Moulton Method (FABMM), nonlinear system, dynamical behavior.

1. Introduction

Chaos theory is a branch of mathematics that focuses on systems that may seem orderly at first glance but actually exhibit chaotic behavior. It involves the study of **nonlinear, dynamic systems** that are highly sensitive to their initial conditions, an effect commonly known as the **butterfly effect [1]**. Despite being **deterministic**, meaning their future behavior is governed by specific rules and not by randomness, these systems are inherently **unpredictable**. Small differences in initial conditions can lead to vastly different outcomes over time, causing trajectories that start close together to diverge exponentially [2]. This theory models the time-dependent behavior of real-world systems using **differential equations**, which describe how the system's variables change over time. These models do not include random variables, yet they can still produce highly complex and seemingly random results. As a relatively new development in **physics and mathematics**, chaos theory has revolutionized how we understand the universe. It offers a powerful framework for analyzing complex, dynamic processes across various fields, from weather systems and fluid dynamics to population growth and financial markets [3].

Dynamical processes are a fundamental part of life and occur in many essential biological functions. Rhythmic behaviors, such as the beating of the heart, breathing, and circadian rhythms, are prime examples of this. To fully understand these periodic processes, mathematical models and numerical simulations are crucial. These tools help us study rhythms across a

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variety of scientific fields, including biology (in areas like rheumatoid arthritis, cancer, asthma, etc.), chemistry, nonlinear optics, fluid dynamics, meteorology, the solar system, and even the heart and brain of living organisms [4]. Fractional calculus, with almost the same history as that of classical calculus (goes back to Leibniz's note in his list to L'Hospital, dated 30 September 1695, in which the meaning of derivative of order one-half was discussed), is a generalization of integer order integration and differentiation, with time memory and long spatial correlation [7]. Applications of this theory can more accurately describe physical phenomena and biochemical reaction processes of memory, heredity, and path dependence. Different from the typical derivative, there are more than six kinds of definitions of fractional derivatives, not mutually equivalent. The Caputo derivative is defined on the basis of the fractional integral and used in this paper. Is analyzed numerically the behavior of a dynamical system, which exhibits high sensitivity to initial conditions, known as chaos [6].

The Rössler system is a three-dimensional nonlinear system that can exhibit chaotic behavior, introduced by Otto Rössler in the 1970s, with an attractor that belongs to the 1-scroll chaotic attractor family [9].

Definition 1.1. For the given function $f(t)$, the Riemann-Liouville fractional integral operator of order $0 < v \le 1$ of a, is defined as [4]:

$$ {}_0I_t^{\alpha} f(t) = D_t^{-v} = \frac{1}{\Gamma(-v)} \int_0^t (t-\tau)^{-v-1} f(\tau) d\tau \tag{1.1} $$

Definition 1.2. For the given function $f(t)$, the Caputo fractional derivative of order $0 < v \le 1$, using formula (1.1), is defined by:

$$ D_t^v f(t) = \frac{1}{\Gamma(1-v)} \int_0^t (t-\tau)^{v-1} f^{(n)}(\tau) d\tau \tag{1.2} $$

where $\Gamma(v) = \int_0^\infty e^{-z} z^{v-1} dz$ is Gamma function! Is chosen Caputo fractional derivative, because it allows initial conditions to be included in the formulation of the problem [7].

Remark 1.1. Fractional Caputo derivative operator, (${}^{RL}D_t^v$ is *R-L* operator), satisfy the following conditions:

- $D_t^v c = 0$
- $D_t^v D_t^{-v} y(t) = {}^{RL}D_t^v D_t^{-v} y(t) = y(t),\ 0 < v \le 1$

Definition 1.3. An *n*-dimensional fractional-order system with derivatives defined in Caputo sense, is described with the (1.3), as follows:

$$ D^v \mathbf{y} = A\mathbf{y}(t) \tag{1.3} $$

with $v = [v_1, v_2, ..., v_n]^T$, $0 < v_i \le 1\ (i = 1, 2, ..., n)$, $\mathbf{y} \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$ [4]. The equilibrium points $E^*(y_1^*, y_2^*, ..., y_n^*)$ of (1.3) are the solutions of the equation $A\mathbf{y}(t) = 0$ [3].

Definition 1.4. The trajectory $y(t) = 0$ of (1.3) is said to be **stable** if for any initial conditions $y_i(t_0) = c_i\ (i = 1, 2, ..., n)$, exist $\varepsilon > 0$, that for any solution $y(t)$ of (1.2) to satisfy the condition $\|y(t)\| < \varepsilon$. $y(t) = 0$ is **asymptotically stable** if it is stable and satisfies $\lim_{t \to \infty} \|y(t)\| = 0$ [4,7].

Theorem 1.1. The equilibrium point $E^*\left(y_1^*, y_2^*, ..., y_n^*\right)$ of the system (1.3) is locally asymptotically stable if all the eigenvalues $\lambda_i \left(i=1,2,...,n\right)$ of the Jacobian matrix $J=\frac{\partial f}{\partial y}$, $f=\left[f_1, f_2, ..., f_n\right]^T$ satisfy the condition [7]:

$$\left|\arg\left(eig\left(J\right)\right)\right| = \left|\arg\left(\lambda_i\right)\right| > v\frac{\pi}{2},\ i=1,2,...,n. \tag{1.4}$$

The equilibrium point is called as non-hyperbolic if $\left|\arg\left(eig\left(J\right)\right)\right| = \left|\arg\left(\lambda_i\right)\right| \neq v\pi$

Theorem 1.2. For the fractional order system (1.3) with $v_i = \frac{n_i}{d_i}$, $\gcd\left(n_i, d_i\right)=1$. Let M be the common multiple of the denominators d_i s. The zero solution of system (1.3) is **globally asymptotically stable** in Lyapunov sense, if all roots λ 's of the equation $\det\left(diag\left(\lambda^{Mv_i}\right)-A\right)=0$ satisfy [3]:

$$\left|\arg\left(\lambda\right)\right| > \frac{\pi}{2M} \tag{1.5}$$

Definition 1.5. For three three-dimensional system, the equilibrium point $E^*\left(y_1^*, y_2^*, y_3^*\right)$ is:

a. **Node:** when all the values $\lambda_i \left(i=1,2,3,...,n\right)$ of $J=\frac{\partial f}{\partial y}$ are real with the same sign;
b. **Saddle:** when all eigenvalues are real and at least one of them is positive and at least one is negative; Saddles are always unstable;
c. **Focus-Node** when it has one real eigenvalue and a pair of complex-conjugate eigenvalues, and all eigenvalues have real parts of the same sign; The equilibrium is stable (unstable) when the sign is negative (positive);
d. **Saddle-Focus:** when it has one real eigenvalue with the sign opposite to the sign of the real part of a pair of complex-conjugate eigenvalues; This type of equilibrium is always unstable; [2]

Definition 1.6. Rössler system is described as [9]:

$$\begin{aligned} D_t^v x\left(t\right) &= -y\left(t\right)-z\left(t\right) \\ D_t^v y\left(t\right) &= x\left(t\right)+ay\left(t\right) \\ D_t^v z\left(x\right) &= b+z\left(t\right)\left(x\left(t\right)-c\right) \end{aligned} \tag{1.6}$$

$x\left(0\right)=1,\ y\left(0\right)=1,\ z\left(0\right)=1,\ \ a\in\left(0,\ 0.6\right),\ b=2,\ c=4$ initial conditions and system parameters, respectively, with fractional order $v=0.9$ and equilibrium points of (1.3), $E_0=\left(0,0,0\right)$ the trivial one and $E_{1/2}=\left(\pm\sqrt{b\left(c+d\right)},\ \pm\sqrt{b\left(c+d\right)},\ c+d\right)$ and Jacobian matrix [7, 11]:

$$J=\begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ z^* & 0 & x^*-c \end{bmatrix} \tag{1.7}$$

2. Numerical method

In recent years, growing interest has emerged across various disciplines of applied science and engineering in the numerical approximation of solutions to fractional-order dynamical systems [8]. Traditional numerical techniques for solving differential equations require adaptation to effectively handle fractional differential equations [11].

Definition 2.1. The problem:

$$D_t^v y(t) = f(t, y(t)), \quad y(0) = y_0, \quad 0 < v \le 1 \tag{2.1}$$

Equation (2.1) is equivalent to the Volterra integral equation given by [3]:

$$y(k) = \sum_{k=0}^{m-1} y_0^{(k)} \frac{t_{n+1}^k}{k!} + \frac{1}{\Gamma(v)} \int_0^t (t-s) f(s, y(s)) ds$$

(2.2)

We will construct the methods, assuming that a solution of (2.1) is sought on some time interval $[0, T]$ arbitrary $0 < v \le 1$ and $f : [0, T] \times D \to \mathbb{R}$, $D \subseteq \mathbb{R}$. The interval $[0, T]$ is divided into l subintervals. Consider an equi-spaced grid with step length h, $t_k = kh,\ k = 0,1,...$ [9]. Let y_k denote the approximated solution at t_k and $y(t_k)$ denote the exact solution of the initial value problem (2.1) [10].

2.1. Adams-Bashforth-Moulton Method: For the considered uniform grid $\{t_n = nh : n = 0,1,..., N\}$, for some integer N and $h = \frac{T}{N},\ 0 < t \le T$. If $\left[D_t^{-v} f(t, y(t))\right]_{t=t_{n+1}}$ is approximated by the fractional trapezoidal formula, the following fractional trapezoidal rule is derived, and equation (2.2) is discretized as:

$$y_h(t_{n+1}) = \sum_{k=0}^{\lceil v \rceil - 1} \frac{t_{n+1}^k}{k!} y_0^{(k)} + \frac{h^v}{\Gamma(v+2)} f(t_{n+1}, y_h^p(t_{n+1})) + \frac{h^v}{\Gamma(v+2)} \sum_{j=0}^{n} a_{j,n+1} f(t_j, y_h(t_j))$$

(2.3)

where $a_{j,n+1}$ are the weight in the corrector (2.3), y_{n+1}^c, given with [6]:

$$a_{j,n+1} = \frac{\Delta t^\alpha}{\Gamma(\alpha+2)} \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^\alpha, & j = 0 \\ (n-j+2)^{\alpha+1} - 2(n-j+1)^{\alpha+1} + (n-j)^{\alpha+1}, & 1 \le j \le n \\ 1, & j = n+1 \end{cases}$$

If $\left[D_t^{-v} f(t, y(t))\right]_{t=t_{n+1}}$ is approximated by the left fractional rectangular formula, the preliminary approximation (2.4), $y_h^p(t_{n+1})$ is called the predictor and is given by:

$$y_h^p(t_{n+1}) = \sum_{k=0}^{\lceil v \rceil - 1} \frac{t_{n+1}^k}{k!} y_0^{(j)} + \frac{1}{\Gamma(v)} \sum_{j=0}^{n} b_{j,n+1} f(t_j, y_h(t_j)) \tag{2.4}$$

where $b_{j,n+1}$ are the weights which depend only on the difference $(j - k)$ because of the convolution structure [10]:

$$b_{j,n+1} = \frac{1}{\Gamma(\alpha+1)} \left[(n-j+1)^\alpha - (n-j)^\alpha\right]$$

The predictor-corrector method, we first use (2.4) to get $y_h^p(t_{n+1})$ (predictor), then we use (2.3) to get $y_h^c(t_{n+1})$ (corrector) by replacing $y_h(t_{n+1})$ with $y_h^p(t_{n+1})$ on the right-hand side of (2.3), which leads to the FABM method [9].

The error in this method is: $\max_{j=0,1,\dots,N}\left|x(t_j)-x_n(t_j)\right| = O(h^p),\ p=\min(2,1+v)$.

3. Comparison and discussions using numerical simulations

Owing to its simplicity, the Rössler system is frequently adopted as a test case for validating chaos control approaches. This section includes three-dimensional simulation results for the fractional-order Rössler system (1.6), conducted with standard parameters $a\in(0,\ 0.6)$, $b=2,\ c=4$, with fractional order $v=0.9$ and initial conditions $x(0)=1,\ y(0)=1,\ z(0)=1$, to analyze its chaotic behavior using FABM (method constructed in Section 2). Is investigated the system's behavior for two selected values of the parameter *a*, with results presented as Case 1 and Case 2. In previous analyses, the dynamics were examined using fixed parameter configurations $a=0.2,\ b=0.2,\ c=5.7$ (system has chaotic behavior) by changing fractional order *v*. Implemented with the aid of Wolfram Mathematica symbolic package, system shows chaotic behavior for $v>0.839$. By varying with parameter $a=(0,\ 0.6)$ and fixed $b=2,\ c=4$, based on FABM, is shown that the system has rich dynamical characteristics with different types of chaos! For *a*=0.01 and *a*=0.56231, chaos changes from periodic cycles to random chaos (deterministic chaos) for the intervals $a=(0,\ 0.2)$ and $a=(0.2,\ 0.6)$ respectively and fractional order $v\in(0,1)$.

Case 1. For $a=0.1,\ b=2,\ c=4$, fractional order $v=0.9$, according to **Theorem 1.2**, stability in Lyapunov sense, the necessary condition is mathematically equivalent to $\frac{\pi}{2M}-\min_i\left\{\left|\arg(\lambda_i)\right|\right\}\geq 0$, where λ_i's are the roots of:

$$\det\left(diag\left(\lambda^{Mv}\quad \lambda^{Mv}\quad \lambda^{Mv}\right)-J_E\right)=0,\ \forall E\in\Omega$$

The equilibrium points: $E_1=(9.89737,\ -98.9737,\ 98.9737)$, $E_2=(6.10263,\ -61.0263,\ 61.0263)$

and eigenvalues: $(2.91942+9.5451i, 2.91942-9.54541i,\ 0.158521)$ and $(1.0351+7.80327i,\ 1.0351-7.80327i,\ 0.132423)$, respectively!

According to **Theorem 1.1**, the equilibrium points are asymptotically stable! For the selected parameters and fractional order, the equation reduces to:

$$-15.7947+100.563\lambda^9-5.99737\lambda^{18}+\lambda^{27}=0$$

with $\min\left\{\left|\arg(\lambda_i)\right|\right\}=0$ and $\frac{\pi}{20}=0.1570796327>0$.

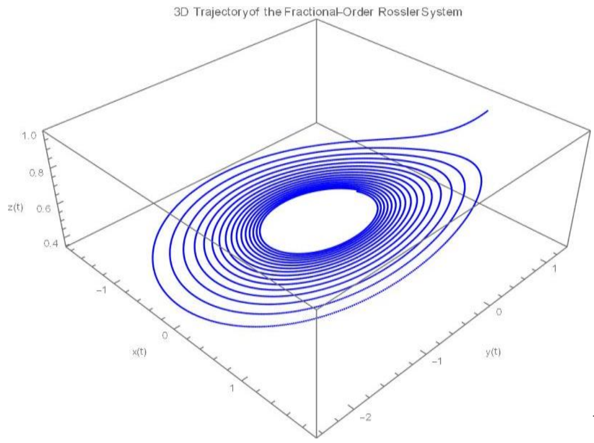
It means, Rössler system with those selected parameters is globally asymptotically stable, with periodic behavior, without needed synchronization and stabilization. See Figure 3.1.

Case 2. For $a=0.5,\ b=2,\ c=4$, fractional order $v=0.9$, according to **Theorem 1.2**, the condition $\left|\arg(\lambda_i)\right|\geq v\frac{\pi}{2}$, is not satisfied $\left|\arg(\lambda_i)\right|=0.9333<0.9\frac{\pi}{2}$ the system is unstable!

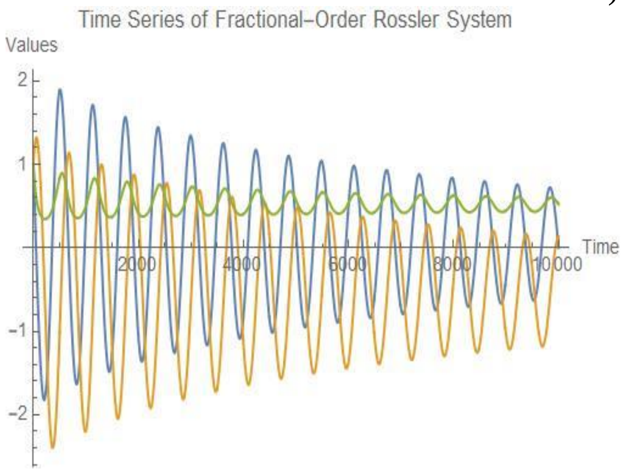
For equilibrium points: $E_1=(9.41421,\ -18.8284,\ 19.8284)$, $E_2=(6.58579,\ -13.1716,\ 13.1716)$

and eigenvalues $(2.55466+3.44917i, 2.55466-3.44917i,\ 0.804881)$,

$(1.0351 + 7.80327i,\, 1.0351 - 7.80327i,\, 0.132423)$, respectively, the stability of the system (1.6) is investigated using firstly **Theorem 1.1**, then by **Theorem 1.2**, taking in considerate non-stability of it.
For the selected parameters and fractional order, the equation reduces to:
$-14.8284 + 22.5355\lambda^9 - 5.9142\lambda^{18} + \lambda^{27} = 0$,
with $\min\{|\arg(\lambda_i)|\} = 0.1037$ and $\dfrac{\pi}{20} = 0.1570796327 > 0.1037$.
It means, Rössler system with those selected parameters satisfy the necessary condition of chaos! See Figure 3.2.

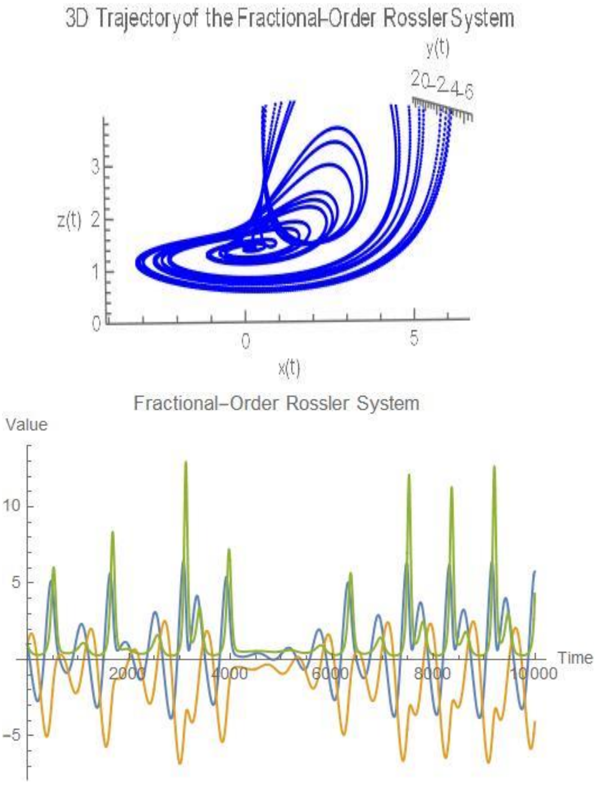


b)



a)

Figure 3.1. a) 3D Trajectory of fractional order Rössler System; b) Time-series of fractional order Rössler system; blue line $x(t)$, orange line $y(t)$ and green line $z(t)$. System exhibit periodic behavior for $a = 0.1,\, b = 2,\, c = 4,\, (x_0, y_0, z_0) = (1, 1, 1)$, and $v = 0.9$.



a) b)

Figure 3.2. a) 3D Trajectory of fractional order Rössler System; b) Time-series of fractional order Rössler system; blue line $x(t)$, orange line, and green line $z(t)$. System exhibit chaotic behavior for $a=0.5,\ b=2,\ c=4,\ (x_0, y_0, z_0)=(1, 1, 1)$ and $v=0.9$.

3.1. Controll of Rössler system using an active control methodolog: An active control law is applied to the incommensurate fractional order Rössler system, using only one input [2,5]! We add an input to the system (1.6) with parameters studied in **Case 2**, then the controlled system is described by:

$$\begin{cases} D_t^{0.9}x(t) = -\left(y(t)+z(t)\right) \\ D_t^{0.9}y(t) = x(t)+0.5y(t) \\ D_t^{0.9}z(t) = z(t)\left(x(t)-4\right)+2+u(t) \end{cases} \tag{3.1}$$

First use the following transformation $u(t)=v(t)-2-z(t)(x(t)-4)$ [3]. Applying this control law to the system (3.1), and selecting the state feedback structure for $v(t)=-k_1x(t)-k_2y(t)-k_3z(t)$, the system (3.1) reduces to (3.2), as follows:

$$\begin{bmatrix} D_t^v x(t) \\ D_t^v y(t) \\ D_t^v z(t) \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0.5 & 0 \\ -k_1 & -k_2 & -k_3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} \tag{3.2}$$

The system is controlled for any value of proper gain, such as the desired poles -1, -2, -3 are located in the stability region [3]! $\det[B \quad AB \quad A^2B]=1\neq 0$ not depend to k_1, k_2, k_3, see [2]. The final controller is $u(t)=14.1769x(t)+8.3014y(t)-6.5z(t)-2+z(t)(4-x(t))$. All three poles satisfy the stability conditions of **Theorem 1.2.**

The single equilibrium point is $E=(0,\ 0,\ 0)$, eigenvalues of Jacobian matrix obtained from formula (1.7) are $(-4,\ 0.25+0.96825\mathrm{i}, 0.25-0.96825\mathrm{i})$, $|\arg(\lambda)|=1.31812<0.9\dfrac{\pi}{2}\approx 1.4372$, the new system (3.1) satisfy the conditions of **Theorem 1.1**.
The characteristic equation from **Theorem 1.2**, is $4-1.\lambda^{9}+3.5\lambda^{18}+\lambda^{27}=0$, with the solutions which satisfy the condition $\min\{|\arg(\lambda_i)|\}=0.146457<\dfrac{\pi}{2M}$ for controlled chaos! This indicates that the proposed controller can linearize the system and has stabilized it for the found value of $a=0.5\in(0.2,\ 0.6)$ where the system exhibits chaotic behavior, see Figure 3.3.

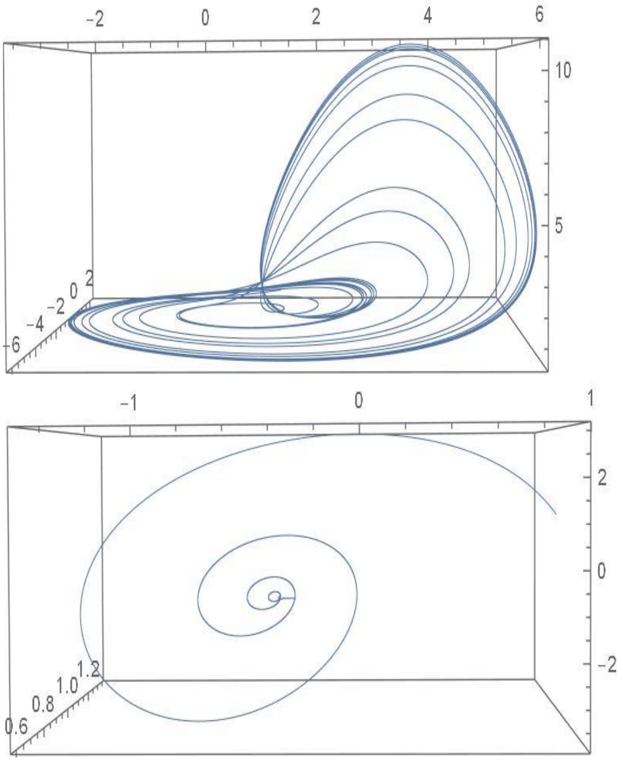


Figure 3.3. a) 3D Trajectory of fractional order Rössler System; b) 3D Trajectory of controlled fractional order Rössler System.

4. Conclusions

In this article, we proposed a modified numerical method of Adams-Bashforth-Moulton, for solving fractional order nonlinear systems! The method was applied to incommensurate fractional order Rössler system, for which the existence of chaotic behavior was analytically and numerically explored! There are found changes in chaotic behaviour of the fractional Rössler system (the system passes from periodic stable behavior to chaotic behavior), firstly, by changing the fractional order and fixing system parameters, and then, by fixing the parameters *b*, *c* and varying $a\in(0,\ 0.6)$, using 3D phase portrait and numerical experiments. With aim to divide the interval of varied *a*, to $a\in(0,\ 0.2)$ and $a\in(0.2,\ 0.6)$ for fractional order *v*=0.9. In all simulations we use FABM, the time of integration is $t\in[0,10000]$, which means that the method should be very fast to deal with it. In the third section, we propose an active control for controlling the chaotic behaviour of the system for *a*=0.5, *b*=2, *c*=4 and *v*=0.9, with two main features: simplicity for practical implementation and the use of a single actuating signal for control.

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