

SHARP INITIAL COEFFICIENT BOUNDS FOR A GENERALIZED JANOWSKI-TYPE SUBCLASS OF STARLIKE FUNCTIONS DEFINED BY SUBORDINATION

Skender AVDIJI

Department of computer science, Public University "Kadri Zeka", Gjilan, Kosova
Email: skender.avdiji@uni-gjilan.net

Abstract

For a function f of the class $\mathcal{A}$ involving an analytic function ϕ using subordination is defined a generalized Janowski-type subclass of normalized analytic functions. For functions belonging to this class, we derive sharp bounds for the initial coefficients $|a_2|$ and $|a_3|$. These bounds are given explicitly in terms of the parameters A and B , as well as the derivatives $\phi'(1)$ and $\phi''(1)$. Several significant special cases are examined, including the classical Janowski starlike class, a square-root-type case, and a case associated with exponential distortion. The sharpness of the obtained results is established by constructing appropriate extremal Schwarz functions

Keywords: univalent functions; differential subordination; coefficient bounds; Janowski functions; Schwarz lemma; starlike functions; sharp estimates.

Introduction and preliminaries

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk, and let $\mathcal{A}$ be the class of a function analytic in $\mathbb{D}$ of the form

$$(1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

A function $f \in \mathcal{A}$ is said to be starlike if

$$(2) \quad \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > 0, z \in \mathbb{D}.$$

and the class of all such function is denoted with S^* .

Lemma 1. (Schwarz Lemma)

If $\omega(z) = \sum_{k=1}^{\infty} c_k z^k$ analytic in $\mathbb{D}$ and satisfy the condition $|\omega(z)| < 1$, then

$$|\omega(z)| \leq |z|, \quad (z \in \mathbb{D}) \text{ and } |c_1| \leq 1$$

are obtained. In particular, $|\omega(z_0)| = |z_0|$, for some $z_0 \in \mathbb{D} \setminus \{0\}$ or $|c_1| = 1$ if and only if $\omega(z) = e^{i\theta} z$ for some θ ($0 \leq \theta < 2\pi$).

An analytic function $\omega: \mathbb{D} \rightarrow \mathbb{C}$ satisfying $\omega(0) = 0$ is called Schwarz function and this function has a Taylor series expansion $\omega(z) = c_1 z + c_2 z^2 + \dots$, with $|\omega(z)| < 1$ for all $z \in \mathbb{D}$.

The standard estimation for a coefficient of a Schwarz function are

$$(3) \quad |c_1| \leq 1, \quad |c_2| \leq 1 - |c_1|^2,$$

both inequalities are sharp

Special cases:

If $c_1 = 0$, then the estimate reduces to

$$|a_2| \leq 1,$$

with equality for

$$\omega(z) = e^{i\theta} z^2.$$

If $|a_1| = 1$, then $w(z) = e^{i\theta} z$ and $a_2 = 0$, which also satisfies the bound sharply.

A standard way to generalize starlike functions is through subordination, because subordination is used in exploring geometric properties of univalent functions. If p and q are analytic in $\mathbb{D}$, then p is subordinate to q , written as

$$p(z) < q(z)$$

If there exists a function $\omega: \mathbb{D} \rightarrow \mathbb{D}$, analytic in $\mathbb{D}$, normalized with $\omega(0) = 0$ with $|\omega(z)| < 1, \forall z \in \mathbb{D}$ such that

$$p(z) = q(\omega(z)), \forall z \in \mathbb{D}.$$

Also the concept of differential subordination plays an important role in geometric properties of analytic functions by embedding them into controlled inequalities. It was formalized by Miller and Mocanu and has become a central tool in modern complex analysis.

Definition 1. Let $\psi: \mathbb{C}^2 \times \mathbb{D} \rightarrow \mathbb{C}$ be a given function and let h be analytic function in $\mathbb{D}$. If an analytic function p satisfies

$$\psi(p(z), zp'(z); z) < h(z), \quad z \in \mathbb{D},$$

then p is said to satisfy a first-order differential subordination.

Janowski functions generalize the classical starlike functions and extend many classical subclasses of univalent functions and by varying of parameters A, B , are obtained different subclasses of univalent functions

The Janowski function is given by

$$(4) \quad J_{A,B}(z) = \frac{1 + A z}{1 + B z}, \quad -1 \leq B < A \leq 1.$$

and the Janowski class of starlike functions $S^*(A, B)$ if

$$(5) \quad \frac{zf'(z)}{f(z)} < \frac{1 + A z}{1 + B z}, \quad z \in \mathbb{D}.$$

Now using this class of a function let us define the class $S^*[A, B; \phi]$.

Let ϕ be analytic in a domain containing 1, with

$$\phi(1) = 1.$$

For $-1 \leq B < A \leq 1$, we define the class

$$(6) \quad S^*[A, B; \phi] = \left\{ f \in \mathcal{A}: \frac{zf'(z)}{f(z)} < \phi\left(\frac{1 + A z}{1 + B z}\right) \right\}$$

Equivalently, $f \in S^*[A, B; \phi]$ if there exists a Schwarz function

$$\omega(z) = c_1 z + c_2 z^2 + \dots$$

such that

$$\frac{zf'(z)}{f(z)} = \phi\left(\frac{1 + A \omega(z)}{1 + B \omega(z)}\right).$$

For a Schwarz function we use the standard estimates

$$|c_1| \leq 1 \text{ and } |c_2| \leq 1 - |c_1|^2.$$

This inequality are sharp.

2. Main result

Theorem 1. Let $f \in S^*[A, B; \phi]$, where

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots$$

Assume that ϕ is analytic at 1, with $\phi(1) = 1$. Then

$$(7) \quad |a_2| \leq |A - B| |\phi'(1)|$$

and

$$(8) \quad |a_3| \leq \frac{1}{2} \max \left\{ |(A-B)\phi'(1)|, \left| (A-B)^2(\phi'(1))^2 - B(A-B)\phi'(1) + \frac{(A-B)^2}{2}\phi''(1) \right| \right\}.$$

Both estimates are sharp.

Proof

Since $f \in S^*[A, B; \phi]$, there exists a Schwarz function

$$\omega(z) = c_1 z + c_2 z^2 + \dots$$

such that

$$\frac{zf'(z)}{f(z)} = \phi \left(\frac{1 + A\omega(z)}{1 + B\omega(z)} \right).$$

Firs, from Janowski function we have the expansion

$$\frac{1 + A\omega(z)}{1 + B\omega(z)} = 1 + (A-B)\omega(z) - B(A-B)\omega(z)^2 + \dots$$

and since

$$\omega(z) = c_1 z + c_2 z^2 + \dots,$$

we obtain

$$\frac{1 + A\omega(z)}{1 + B\omega(z)} = 1 + (A-B)c_1 z + [(A-B)c_2 - B(A-B)c_1^2]z^2 + \dots,$$

now if we expand ϕ around 1

$$\phi(\omega) = 1 + \phi'(1)(\omega - 1) + \frac{\phi''(1)}{2}(\omega - 1)^2 + \dots.$$

Therefore

$$\begin{aligned} \phi \left(\frac{1 + A\omega(z)}{1 + B\omega(z)} \right) &= 1 + (A-B)\phi'(1)c_1 z \\ &+ \left[(A-B)\phi'(1)c_2 - B(A-B)\phi'(1)c_1^2 + \frac{(A-B)^2}{2}\phi''(1)c_1^2 \right] z^2 + \dots \end{aligned}$$

On the other hand,

$$\frac{zf'(z)}{f(z)} = 1 + a_2 z + (2a_3 - a_2^2)z^2 + \dots.$$

Comparing coefficients gives

$$a_2 = (A-B)\phi'(1)c_1.$$

Hence

$$|a_2| \leq |A-B||\phi'(1)|$$

For the second coefficient ,

$$2a_3 - a_2^2 = (A-B)\phi'(1)c_2 - B(A-B)\phi'(1)c_1^2 + \frac{(A-B)^2}{2}\phi''(1)c_1^2.$$

Since

$$a_2^2 = (A-B)^2(\phi'(1))^2 c_1^2,$$

we obtain

$$a_3 = \frac{(A-B)\phi'(1)}{2} c_2 + \frac{1}{2} \left[(A-B)^2(\phi'(1))^2 - B(A-B)\phi'(1) + \frac{(A-B)^2}{2}\phi''(1) \right] c_1^2.$$

If we put

$$(9) \quad M = (A - B)\phi'(1),$$

and

$$(10) \quad N = (A - B)^2(\phi'(1))^2 - B(A - B)\phi'(1) + \frac{(A - B)^2}{2}\phi''(1),$$

then

$$a_3 = \frac{1}{2}Mc_2 + \frac{1}{2}Nc_1^2.$$

Using the inequality

$$|c_2| \leq 1 - |c_1|^2,$$

we obtain

$$|a_3| \leq \frac{1}{2}|M|(1 - |c_1|^2) + \frac{1}{2}|N||c_1|^2$$

Let

$$x = |c_1|^2, \quad 0 \leq x \leq 1.$$

Then

$$|a_3| \leq \frac{1}{2}[|M|(1 - x) + |N|x].$$

Since the right-hand side is linear in x , its maximum on $[0,1]$ is attained at an endpoint. Hence

$$|a_3| \leq \frac{1}{2}\max\{|M|, |N|\}.$$

Thus

$$|a_3| \leq \frac{1}{2}\max\left\{|(A - B)\phi'(1)|, \left|(A - B)^2(\phi'(1))^2 - B(A - B)\phi'(1) + \frac{(A - B)^2}{2}\phi''(1)\right|\right\}.$$

This proves the desired estimate.

Now let's proof the sharpnes.

For $|a_2|$, equality holds when

$$|c_1| = 1.$$

By Schwarz lemma this happens when

$$\omega(z) = e^{i\theta}z.$$

Now let's construct the extremal function,

$$\frac{zf'_\theta(z)}{f_\theta(z)} = \phi\left(\frac{1 + Ae^{i\theta}z}{1 + Be^{i\theta}z}\right)$$

Since

$$z \frac{d}{dz} \log \frac{f(z)}{z} = \frac{zf'(z)}{f(z)} - 1$$

we can write

$$\frac{d}{dz} \log \frac{f(z)}{z} = \frac{\phi\left(\frac{1 + Ae^{i\theta}z}{1 + Be^{i\theta}z}\right) - 1}{z}$$

Integrating from 0 to z

$$\log \frac{f(z)}{z} = \int_0^z \frac{\phi\left(\frac{1 + Ae^{i\theta}t}{1 + Be^{i\theta}t}\right) - 1}{t} dt$$

or

$$f(z) = z e^{\int_0^z \frac{\phi\left(\frac{1+Ae^{i\theta}t}{1+Be^{i\theta}t}\right)-1}{t} dt}$$

For $|a_3|$, the sharpness argument is based on the fact that a_3 depends on the first two Schwarz coefficients of ω .

From

$$|a_3| \leq |N|x + |M|(1-x), \quad x = |c_1|^2, \quad 0 \leq x \leq 1.$$

So the problem reduces to maximizing the linear function

$$G(x) = |N|x + |M|(1-x), \quad 0 \leq x \leq 1.$$

$$G'(x) = |N| - |M|$$

There are two cases.

Case 1. $|N| > |M|$

$G'(x) > 0$, $G(x)$ is increasing $\Rightarrow |N| - |M| > 0 \Rightarrow |N| > |M|$, the maximum occurs at $x = 1$,

$$|c_1|^2 = x = 1$$

By the equality case of Schwarz's lemma,

$$\omega(z) = e^{i\theta} z,$$

that is $c_1 = e^{i\theta}$, $c_2 = 0$

Hence

$$a_3 = Ne^{2i\theta}, \Rightarrow |a_3| = |N|$$

Thus, when $|N| > |M|$, equality is obtained by taking

$$\omega(z) = e^{i\theta} z.$$

Case 2. $|N| < |M|$

$G'(x) < 0$, $G(x)$ is decreasing $\Rightarrow |N| - |M| < 0 \Rightarrow |N| < |M|$, the maximum occurs at $x = 0$, this mean $c_1 = 0$

To get equality in the Schwarz estimate, we take

$$\omega(z) = e^{i\theta} z^2$$

For this choice,

$$c_1 = 0, \quad c_2 = e^{i\theta}.$$

Hence

$$a_3 = Me^{i\theta} \text{ so } |a_3| = |M|$$

Thus when $|N| < |M|$, equality is obtained by taking

$$\omega(z) = e^{i\theta} z^2.$$

Case 3. $|M|=|N|$

$$|a_3| \leq |N|x + N(1-x) = |N|$$

Therefor

$$|a_3| \leq |N| = M$$

both extremal choices work

Thus the bound for $|a_3|$ is sharp.

Corollary 1 (Classical Janowski Starlike Class)

Let $\phi(w) = w$.

Then

$$S^*[A, B; \phi] = S^*[A, B],$$

the usual Janowski starlike class.

If $f \in S^*[A, B]$, then

$$|a_2| \leq A - B,$$

and

$$|a_3| \leq \frac{A-B}{2} \max\{1, |A-2B|\}.$$

Proof

Since

$$\phi(w) = w,$$

We have

$$\phi'(1) = 1, \quad \phi''(1) = 0.$$

Substituting these into the main theorem gives

$$|a_2| \leq A - B,$$

Also,

$$N = (A - B)^2 - B(A - B) = (A - B)(A - 2B)$$

Therefor

$$|a_3| \leq \frac{1}{2} \max\{A - B, |(A - B)(A - 2B)|\}.$$

Since $A - B > 0$, we get

$$|a_3| \leq \frac{A-B}{2} \max\{1, |A-2B|\}$$

Corollary 2 (Classical Starlike Functions) Let $A = 1, B = -1, \phi(w) = w$. Then

$$S^*[1, -1; \phi] = S^*,$$

the classical class of starlike functions. If

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots \in S^*$$

Then

$$|a_2| \leq 2 \text{ and } |a_3| \leq 3.$$

Proof

From Corollary 1,

$$|a_2| \leq A - B = 1 - (-1) = 2$$

Also

$$A - 2B = 3$$

Thus

$$|a_3| \leq \frac{2}{2} \max\{1, 3\} = 3$$

These are the well-known sharp coefficient bounds for the classical starlike class.

Corollary 3. (The case $\phi(w) = w^\mu$).

Let

$$\phi(w) = w^\mu, \mu > 0$$

where the principal branch is taken near $w = 1$.

If $f \in S^*[A, B; w^\mu]$,

$$|a_2| \leq \mu(A - B),$$

$$|a_3| \leq \frac{\mu(A-B)}{2} \max\left\{1, \left|-B + \frac{(A-B)(3\mu-1)}{2}\right|\right\}.$$

Proof

For

$$\phi(w) = w^\mu,$$

we have

$$\phi'(1) = \mu, \quad \phi''(1) = \mu(\mu - 1)$$

Therefor

$$|a_2| \leq \mu(A - B).$$

Moreover

$$N = (A - B)^2 \mu^2 - B(A - B)\mu + \frac{(A - B)^2}{2} \mu(\mu - 1).$$

Hence

$$N = \mu(A - B) \left[-B + \frac{(A - B)(3\mu - 1)}{2} \right].$$

Therefor

$$|a_3| \leq \frac{1}{2} \max \left\{ \mu(A - B), \mu(A - B) \left| -B + \frac{(A - B)(3\mu - 1)}{2} \right| \right\}$$

Thus,

$$|a_3| \leq \frac{\mu(A - B)}{2} \max \left\{ 1, \left| -B + \frac{(A - B)(3\mu - 1)}{2} \right| \right\}$$

Corollary 4. (Square-Root Subclass)

Let $\phi(w) = \sqrt{w}$. Then $\mu = \frac{1}{2}$ in Corollary 3.

If

$$f \in S^*[A, B; \sqrt{w}],$$

Then

$$|a_2| \leq \frac{A - B}{2},$$

And

$$|a_3| \leq \frac{A - B}{4} \max \left\{ 1, \left| -B + \frac{A - B}{4} \right| \right\}.$$

In particular, for $A = 1$ and $B = 0$.

$$|a_2| \leq \frac{1}{2}, \quad |a_3| \leq \frac{1}{4}.$$

Corollary 5: (Exponential-Type Subclass). Let

$$\phi(w) = e^{w-1}.$$

If

$$f \in S^*[A, B; e^{w-1}]$$

Then

$$|a_2| \leq A - B,$$

And

$$|a_3| \leq \frac{A - B}{2} \max \left\{ 1, \left| -B + \frac{3}{2}(A - B) \right| \right\}.$$

Proof

For

$$\phi(w) = e^{w-1},$$

We have

$$\phi'(1) = 1, \quad \phi''(1) = 1.$$

Therefor

$$|a_2| \leq A - B.$$

Also,

$$N = (A - B)^2 \left[-B + \frac{3}{2}(A - B) \right].$$

Therefor

$$|a_3| \leq \frac{A - B}{2} \max \left\{ 1, \left| -B + \frac{3}{2}(A - B) \right| \right\}.$$

Corollary 6 (Exponential Case with $A = 1, B = -1$)

Let

$$A = 1, \quad B = -1, \quad \phi(w) = e^{w-1}.$$

If

$$f \in S^*[1, -1; e^{w-1}],$$

Then

$$\begin{aligned} |a_2| &\leq 2; \\ |a_3| &\leq 4 \end{aligned}$$

Corollary 7. (The case $B=0$ and the case $A=1, B=0$)

The case $B=0$.

Then the class is determined by

$$\frac{zf'(z)}{f(z)} < \phi(1 + Az)$$

If $f \in S^*[A, 0, \phi]$,

Then

$$|a_2| \leq |A||\phi'(w)|$$

And

$$|a_3| \leq \frac{1}{2} \max \left\{ |A\phi'(w)|, \left| A^2(\phi'(1))^2 + \frac{A^2}{2}\phi''(w) \right| \right\}$$

The case $A=1, B=0$.

Then the class is determined by

$$\frac{zf'(z)}{f(z)} < \phi(1 + z)$$

If $f \in S^*[1, 0; \phi]$

Then

$$|a_2| \leq |\phi'(1)|,$$

And

$$|a_3| \leq \frac{1}{2} \max \{ |\phi'(1)|, |(\phi'(1))^2 + \frac{1}{2}\phi''(1)| \}$$

Below we have presented a table for the bound of $|a_2|$ and $|a_3|$ where are given special cases for the function $\phi(w)$ and the parameters $-1 \leq B < A \leq 1$

Example	$\phi(w)$	(A, B)	$ a_2 , bound$	$ a_3 , bound$
1	w	$(\frac{1}{2}, -\frac{1}{2})$	1	$\frac{3}{4}$
2	w	$(1, -1)$	2	3
3	w^2	$(1, 0)$	2	$\frac{5}{2}$
4	$\sqrt{w}$	$(1, 0)$	$\frac{1}{2}$	$\frac{1}{4}$
5	e^{w-1}	$(1, 0)$	1	$\frac{3}{4}$
7	e^{w-1}	$(1, -1)$	2	4
8	w^2	$(\frac{1}{2}, 0)$	1	$\frac{5}{8}$
9	$1 + \sin(w - 1)$	$(1, 0)$	1	$\frac{1}{2}$

Conclusion

In this paper, we considered a generalized Janowski-type subclass of starlike functions defined by the differential subordination

$$\frac{zf'(z)}{f(z)} < \phi\left(\frac{1 + Az}{1 + Bz}\right), -1 \leq B < A \leq 1, \quad z \in \mathbb{D}$$

where ϕ is analytic in a neighborhood of 1 and satisfies $\phi(1) = 1$. By using the power-series expansion of the logarithmic derivative, the Janowski function, and the standard coefficient estimates for Schwarz functions, we obtained sharp upper bounds for the initial coefficients $|a_2|$ and $|a_3|$. The main estimates are expressed explicitly in terms of the parameters A, B , and the quantities $\phi'(1)$ and $\phi''(1)$, which shows clearly how the choice of the function ϕ influences the coefficient structure of the class.

The obtained bounds are

$$|a_2| \leq |A - B||\phi'(1)|$$

and

$$|a_3| \leq \frac{1}{2} \max \left\{ |(A - B)\phi'(1)|, \left| (A - B)^2(\phi'(1))^2 - B(A - B)\phi'(1) + \frac{(A - B)^2}{2}\phi''(1) \right| \right\}.$$

The sharpness of these estimates are verified by choosing suitable extremal Schwarz functions, particularly

$$\omega(z) = e^{i\theta}z$$

and

$$\omega(z) = e^{i\theta}z^2.$$

Several special cases are also given. In particular, the general theorem reduces to the classical Janowski starlike class when $\phi(w) = w$, and to the classical starlike class when $A=1, B=-1$.

Also some example are given in a table form for special cases of choice $\phi(w)$ and A and B and the result in bounds.

The results of this paper may be extended in several directions. One natural continuation is to study the Fekete–Szegő functional for function $S^*[A, B; \phi]$. Another possible direction is the investigation of higher-order coefficients such as $|a_4|$, $|a_5|$, and inverse coefficients. It would also be interesting to study Hankel determinants, logarithmic coefficients, and radius problems for this generalized class.

References

- [1]. W. Janowski, "Some extremal problems for certain families of analytic functions," *Annales Polonici Mathematici*, 28, 1973, 297–326.
- [2]. S. S. Miller and P. T. Mocanu, *Differential Subordinations: Theory and Applications*, Marcel Dekker, New York, 2000.
- [3]. W. Ma and D. Minda, "A unified treatment of some special classes of univalent functions," in *Proceedings of the Conference on Complex Analysis*, International Press, 1992, 157–169.
- [4]. P. L. Duren, *Univalent Functions*, Springer-Verlag, New York, 1983.
- [5]. W. Goodman, *Univalent Functions*, Vols. I–II, Mariner Publishing, Tampa, Florida, 1983.
- [6]. Avdiji, S. and Tuneski, N., 2020. *Sufficient conditions for starlikeness using subordination method*. *Adv Math Sci J*, 9(12), pp.10707-10716.
- [7]. Avdiji, S., & Tuneski, N. *Sufficient condition over the certain differential subordination*
- [8]. Avdiji, S., & Tuneski, N. (2020). *Application of differential subordination for sufficient condition for univalence*. *Advances in Mathematics: Scientific Journal*, 9(3), 1341-1348.
- [9]. Shiraishi, H., & Hayami, T. (2013). *Coefficient estimates for Schwarz functions*. arXiv preprint arXiv:1302.6916. <https://doi.org/10.48550/arXiv.1302.6916>