

ON ALMOST AND ALMOST STATISTICAL CONVERGENCE IN CONE METRIC SPACES

Fatih NURAY

Afyon Kocatepe University, Afyonkarahisar, Turkey
fnuray@aku.edu.tr

Abstract

We introduce the notions of almost convergence and almost statistical convergence in cone metric spaces, combining cone-ordered distances with shift-uniform Cesàro averaging and density methods.

We establish basic structural properties including uniqueness of limits and order-dominance invariance, and show that almost convergence implies almost statistical convergence under a natural norm-control condition. We also provide examples showing that the converse fails in general.

We further study almost statistical sequential compactness and obtain boundedness and closedness properties. In addition, we introduce almost statistically Cauchy sequences and a corresponding notion of completeness. We prove that in complete cone metric spaces, every almost statistically Cauchy sequence admits a convergent subsequence, and hence has an ordinary limit point.

These results provide a unified framework for shift-uniform convergence in cone metric spaces and suggest directions for further research.

Keywords: Cone metric space, almost convergence, almost statistical convergence, almost statistical compactness
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1 Introduction

Cone metric spaces, introduced by Huang and Zhang [7], extend the classical metric framework by replacing real-valued distances with vector-valued distances ordered by a cone. This setting has proved to be a useful tool in nonlinear analysis, particularly in fixed point theory and related areas. In a cone metric space, the distance takes values in a Banach space E equipped with a cone P , and convergence is defined via the strong order relation $\ll$ induced by the interior $\text{int}P$. This ordered structure naturally raises the question of how classical notions of convergence and summability can be adapted to this more general framework.

On the other hand, statistical convergence, introduced independently by Fast [2] and Steinhaus [14], provides a flexible weakening of ordinary convergence based on the asymptotic density of exceptional sets. Since its introduction, statistical convergence and its variants have played an important role in summability theory and sequence space theory. A key direction in this area has been the incorporation of *shift-invariance* or *uniformity under translations*, which leads naturally to almost convergence in the sense of Lorentz [12]. Almost convergence can be characterized by the uniform convergence of Cesàro means with respect to shifts, making it a stronger and more stable notion than classical convergence.

Motivated by these developments, the purpose of this paper is to introduce and study the concepts of *almost convergence* and *almost statistical convergence* in the setting of cone metric spaces. These notions combine three different structures: the cone-ordered geometry of the space, Cesàro-type averaging, and density-based asymptotics with shift-uniform control. In particular, almost

statistical convergence strengthens classical statistical convergence by requiring uniformity with respect to shifts, thus aligning it with Lorentz-type almost convergence in an ordered vector setting. After recalling basic definitions in Section 2, we introduce in Section 3 the notions of almost convergence and almost statistical convergence in cone metric spaces. We establish fundamental properties such as uniqueness of limits and order-dominance invariance. Furthermore, we prove that almost convergence implies almost statistical convergence under a natural norm-control condition. We also provide examples showing that the converse implication does not hold, demonstrating that almost statistical convergence strictly extends almost convergence.

In Section 4, we investigate almost statistical sequential compactness. We show that both sequential compactness and almost sequential compactness imply almost statistical sequential compactness, and we derive basic permanence properties such as boundedness and closedness.

Section 5 is devoted to almost statistical completeness. We introduce the notion of almost statistically Cauchy sequences, which is formulated in a shift-uniform manner. Due to the presence of this shift-uniform structure, the classical equivalence between convergence and Cauchy conditions does not directly extend to this setting. Instead, we establish subsequential convergence results: in particular, we prove that every almost statistically convergent sequence admits a convergent subsequence, and that in a complete cone metric space, every almost statistically Cauchy sequence has a subsequence which is Cauchy (and hence convergent) in the usual sense. This shows that almost statistical completeness provides a natural, though weaker, analogue of classical completeness.

Throughout the paper, our approach is based on the nonlinear scalarization ξ_e , which allows us to reduce cone-valued estimates to real-valued ones while preserving the essential order structure. We emphasize that certain intermediate results rely on the normality of the cone; these assumptions are explicitly stated when required.

The results obtained in this paper provide a unified framework for studying shift-uniform convergence phenomena in cone metric spaces. They also open several directions for further research, including strengthening completeness-type results, investigating different cone geometries (such as regular or solid cones), and extending the theory to operator sequences, double sequences, and other summability methods.

2 Preliminaries

Definition 1 Let E be a real Banach space and $P \subset E$ be a cone (i.e., P is closed, nonempty, $P + P \subset P$, $\lambda P \subset P$ for $\lambda \geq 0$, and $P \cap (-P) = \{0\}$). The partial ordering induced by P is defined by

$$x \leq y \iff y - x \in P.$$

We write $x \ll y$ if $y - x \in \text{int}P$. Throughout this paper, we assume $\text{int}P \neq \emptyset$.

Definition 2 Let X be a nonempty set. A mapping $d: X \times X \rightarrow E$ is called a cone metric if

1. $0 \leq d(x, y)$ and $d(x, y) = 0$ iff $x = y$,
2. $d(x, y) = d(y, x)$,
3. $d(x, z) \leq d(x, y) + d(y, z)$.

Then (X, d) is called a cone metric space.

Remark 1 Convergence in a cone metric space is defined as follows: (x_n) converges to x (denoted $x_n \rightarrow x$) if for every $c \in \text{int}P$, there exists N_c such that $d(x_n, x) \ll c$ for all $n \geq N_c$.

3 Almost Convergence and Almost Statistical Convergence

The foundational roots of almost convergence lie in the theory of Banach limits—non-constructive linear functionals on the space of bounded sequences that extend the limit functional and satisfy shift-invariance. A sequence is said to be almost convergent if all its Banach limits coincide. The fundamental characterization provided by Lorentz [12] states that a sequence (x_n) is almost convergent to L if and only if its Cesàro averages vanish uniformly with respect to the shift parameter m ; that is, $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N x_{k+m} = L$ uniformly for $m \in \mathbb{N}$. In our current work, we utilize this shift-uniform characterization as the primary template for defining almost convergence in cone metric spaces. By doing so, we essentially seek the cone-metric analogue of a sequence whose 'average' vector-valued distance becomes uniformly negligible, mirroring the stability of Banach limits in an ordered vector space setting.

Definition 3 Let (X, d) be a cone metric space. A sequence (x_n) in X is said to be almost convergent to $x \in X$ if for every $c \in \text{int}P$ there exists $N_c \in \mathbb{N}$ such that for all $N \geq N_c$ and for all $m \in \mathbb{N}$,

$$\frac{1}{N} \sum_{k=1}^N d(x_{k+m}, x) \ll c.$$

In this case we write $x_n \xrightarrow{ac} x$.

Remark 2 Every convergent sequence is almost convergent, but the converse does not hold in general.

Example 1 Let $X = \mathbb{R}$ and let $E = \mathbb{R}^2$ with the cone $P = \{(u, v): u \geq 0, v \geq 0\}$. Define $d(x, y) = (|x - y|, |x - y|^2)$. Consider the sequence

$$x_n = \begin{cases} 1, & n = j^2, \\ 0, & \text{otherwise.} \end{cases}$$

For any m, N , the number of squares in $\{m + 1, \dots, m + N\}$ is at most $\sqrt{N} + 1$. Hence,

$$\frac{1}{N} \sum_{k=1}^N d(x_{k+m}, 0) = \left(\frac{S_{N,m}}{N}, \frac{S_{N,m}}{N} \right).$$

Here $S_{N,m} = |\{k \in \{1, \dots, N\}: k + m \text{ is a perfect square}\}| = \sum_{k=1}^N \mathbf{1}_{\{j^2: j \in \mathbb{N}\}}(k + m)$.

Thus for any $c = (c_1, c_2) \gg (0, 0)$, choosing N_c sufficiently large gives $x_n \xrightarrow{ac} 0$.

Definition 4 Let (X, d) be a cone metric space. A sequence (x_n) in X is said to be almost statistically convergent to $x \in X$ if for every $c \in \text{int}P$,

$$\limsup_{N \rightarrow \infty} \frac{1}{N} |\{k \leq N: d(x_{k+m}, x) \not\ll c\}| = 0.$$

In this case we write $x_n \xrightarrow{asc} x$.

Theorem 1 The limit of an almost convergent (resp. almost statistically convergent) sequence is unique.

Proof. Suppose (x_n) is almost convergent to both x and y . For any $c \in \text{int}P$, choose N_c such that for all $N \geq N_c$ and all m , $\frac{1}{N} \sum_{k=1}^N d(x_{k+m}, x) \ll c/2$ and $\frac{1}{N} \sum_{k=1}^N d(x_{k+m}, y) \ll c/2$. Then by the triangle inequality,

$$d(x, y) \leq \frac{1}{N} \sum_{k=1}^N d(x_{k+m}, x) + \frac{1}{N} \sum_{k=1}^N d(x_{k+m}, y) \ll c.$$

Since c was arbitrary, $d(x, y) = 0$ and $x = y$. The proof for almost statistical convergence is analogous.

Theorem 2 (Order-dominance invariance) *Let (x_n) and (y_n) be sequences such that $d(x_n, a) \leq d(y_n, a)$ for all n . If (y_n) is almost statistically convergent to a , then (x_n) is also almost statistically convergent to a .*

Proof. Fix $c \in \text{int}P$ and define $A_{N,m}(c) = \{k \leq N: d(x_{k+m}, a) \ll c\}$, $B_{N,m}(c) = \{k \leq N: d(y_{k+m}, a) \ll c\}$. We claim $A_{N,m}(c) \subseteq B_{N,m}(c)$. Indeed, if $k \in A_{N,m}(c)$ but $k \notin B_{N,m}(c)$, then $d(y_{k+m}, a) \ll c$. Since $d(x_{k+m}, a) \leq d(y_{k+m}, a)$, we have $c - d(x_{k+m}, a) = (c - d(y_{k+m}, a)) + (d(y_{k+m}, a) - d(x_{k+m}, a)) \in \text{int}P + P \subset \text{int}P$, so $d(x_{k+m}, a) \ll c$, a contradiction. Hence $|A_{N,m}(c)| \leq |B_{N,m}(c)|$ for all N, m , giving

$$\sup_m \frac{|A_{N,m}(c)|}{N} \leq \sup_m \frac{|B_{N,m}(c)|}{N} \rightarrow 0.$$

Thus (x_n) is almost statistically convergent to a .

Lemma 1 (Nonlinear scalarization and basic properties) *Fix $e \in \text{int}P$ and define*

$$\xi_e(y) := \inf\{\lambda > 0: y \ll \lambda e\} \quad (y \in E).$$

Then the following hold:

- (i) $\xi_e(y) \geq 0$ for $y \in P$, and $\xi_e(0) = 0$.
- (ii) (Monotonicity) If $y_1 \leq y_2$ then $\xi_e(y_1) \leq \xi_e(y_2)$.
- (iii) (Positive homogeneity) $\xi_e(\alpha y) = \alpha \xi_e(y)$ for all $\alpha \geq 0$ and $y \in P$.
- (iv) (Subadditivity on P) $\xi_e(y_1 + y_2) \leq \xi_e(y_1) + \xi_e(y_2)$ for all $y_1, y_2 \in P$.
- (v) For every $c \in \text{int}P$ there exists $\varepsilon > 0$ such that $\varepsilon e \ll c$. Moreover, for every $\varepsilon > 0$ we have $\varepsilon e \in \text{int}P$.

Proof. (i)–(iii) are immediate from the definition and the fact that P is a cone and $\text{int}P$ is stable under positive scaling. (iv) Fix $y_1, y_2 \in P$ and $\eta > 0$. Choose $\lambda_i > \xi_e(y_i)$ such that $y_i \ll \lambda_i e$ for $i = 1, 2$. Then $(\lambda_1 e - y_1), (\lambda_2 e - y_2) \in \text{int}P$, hence

$$(\lambda_1 + \lambda_2)e - (y_1 + y_2) = (\lambda_1 e - y_1) + (\lambda_2 e - y_2) \in \text{int}P,$$

so $y_1 + y_2 \ll (\lambda_1 + \lambda_2)e$. Taking infimum over $\lambda_i \downarrow \xi_e(y_i)$ yields (iv). (v) Since $\text{int}P$ is open and contains c , there is $t > 0$ with $c - te \in \text{int}P$, i.e. $te \ll c$.

Lemma 2 *Define $\rho: X \times X \rightarrow [0, \infty)$ by $\rho(u, v) := \xi_e(d(u, v))$. Then ρ is an ordinary (real-valued) metric on X . Moreover, for any $c \in \text{int}P$ there exists $\varepsilon > 0$ such that*

$$d(u, v) \ll c \iff \rho(u, v) < \varepsilon, \quad \text{and} \quad d(u, v) \ll \varepsilon e \iff \rho(u, v) < \varepsilon.$$

Proof. Symmetry and $\rho(u, v) = 0 \Rightarrow u = v$ follow from the corresponding properties of d and Lemma 1(i). For the triangle inequality, note that $d(u, w) \leq d(u, v) + d(v, w)$ with all terms in P . By monotonicity and subadditivity on P ,

$$\rho(u, w) = \xi_e(d(u, w)) \leq \xi_e(d(u, v) + d(v, w)) \leq \xi_e(d(u, v)) + \xi_e(d(v, w)) = \rho(u, v) + \rho(v, w).$$

The equivalences follow directly from the definition of ξ_e and Lemma 1(v).

Lemma 3 (Real almost convergence implies real almost statistical convergence) *Let (a_n) be a nonnegative real sequence. If*

$$\limsup_{N \rightarrow \infty} \sup_{m \in \mathbb{N}} \frac{1}{N} \sum_{k=1}^N a_{k+m} = 0,$$

then for every $\varepsilon > 0$,

$$\limsup_{N \rightarrow \infty} \sup_{m \in \mathbb{N}} \frac{1}{N} |\{k \leq N: a_{k+m} \geq \varepsilon\}| = 0.$$

Proof. Fix $\varepsilon > 0$. For every N, m we have

$$\frac{1}{N} \sum_{k=1}^N a_{k+m} \geq \frac{1}{N} \sum_{\substack{k \leq N \\ a_{k+m} \geq \varepsilon}} a_{k+m} \geq \varepsilon \cdot \frac{1}{N} |\{k \leq N: a_{k+m} \geq \varepsilon\}|.$$

Taking $\sup_m$ and then $N \rightarrow \infty$ gives the claim.

Lemma 4 *Let (X, d) be a cone metric space with a normal cone P . If (x_n) is almost convergent to x , then the real sequence*

$$b_n := \|d(x_n, x)\|$$

satisfies

$$\limsup_{N \rightarrow \infty} \sup_{m \in \mathbb{N}} \left\| \frac{1}{N} \sum_{k=1}^N d(x_{k+m}, x) \right\| = 0.$$

Proof. Assume $x_n \xrightarrow{ac} x$. Let $K > 0$ be the normal constant of P . Fix $\varepsilon > 0$. Choose $c \in \text{int}P$ such that

$$K \|c\| < \varepsilon.$$

Then for all sufficiently large N and all m ,

$$\frac{1}{N} \sum_{k=1}^N d(x_{k+m}, x) \ll c,$$

hence,

$$0 \leq \frac{1}{N} \sum_{k=1}^N d(x_{k+m}, x) \leq c.$$

By normality,

$$\left\| \frac{1}{N} \sum_{k=1}^N d(x_{k+m}, x) \right\| \leq K \|c\| < \varepsilon.$$

Taking supremum over m gives the result.

Theorem 3 *Let (X, d) be a cone metric space with a normal cone P . If (x_n) is almost convergent to x , then (x_n) is almost statistically convergent to x .*

Proof. Fix $e \in \text{int}P$ and define $\rho(u, v) := \xi_e(d(u, v))$.

By Lemma 4, we have

$$\limsup_{N \rightarrow \infty} \sup_m \frac{1}{N} \sum_{k=1}^N \rho(x_{k+m}, x) = 0.$$

Fix $c \in \text{int}P$. Since $\text{int}P$ is absorbing, there exists $\lambda > 0$ such that

$$\lambda e \ll c.$$

Set $\varepsilon = \lambda$.

For each N and m , define

$$A_{N,m}(c) = \{k \leq N: d(x_{k+m}, x) \ll c\}.$$

If $d(x_{k+m}, x) \ll c$, then necessarily $\rho(x_{k+m}, x) \geq \varepsilon$. Hence

$$A_{N,m}(c) \subseteq \{k \leq N: \rho(x_{k+m}, x) \geq \varepsilon\}.$$

Thus,

$$\frac{|A_{N,m}(c)|}{N} \leq \frac{1}{N} \sum_{k=1}^N \frac{\rho(x_{k+m}, x)}{\varepsilon}.$$

Taking supremum over m gives

$$\sup_m \frac{|A_{N,m}(c)|}{N} \leq \frac{1}{\varepsilon} \sup_m \frac{1}{N} \sum_{k=1}^N \rho(x_{k+m}, x).$$

Passing to the limit as $N \rightarrow \infty$, we obtain

$$\limsup_{N \rightarrow \infty} \sup_m \frac{|A_{N,m}(c)|}{N} = 0.$$

Therefore $(x_n) \xrightarrow{asc} x$.

Example 2 (Almost statistically convergent but not almost convergent) Let $X = \mathbb{R}$ and $E = C[0,1]$ with the cone $P = \{f: f(t) \geq 0 \ \forall t\}$. Define $d(x, y)(t) = t|x - y| + t^2|x - y|^2$. Consider

$$x_n = \begin{cases} 2^j, & n = 2^j, \\ 0, & \text{otherwise.} \end{cases}$$

Claim 1: (x_n) is almost statistically convergent to 0.

Fix $c \in \text{int}P$. Since c is an interior point, there exists $\delta > 0$ such that $c(t) > \delta$ for all $t \in [0,1]$. For $n = 2^j$, we have $d(x_n, 0)(1/2) = 2^{j-1} + 4^{j-1}$, so for sufficiently large j , $d(x_{2^j}, 0) \ll c$. The number of powers of 2 in any interval of length N is at most $1 + \log_2(N + 1)$. Hence,

$$\sup_{m \in \mathbb{N}} \frac{|\{k \leq N: d(x_{k+m}, 0) \ll c\}|}{N} \leq \frac{1 + \log_2(N + 1)}{N} \rightarrow 0.$$

Claim 2: (x_n) is not almost convergent to 0.

Take $c \equiv 1 \in \text{int}P$. If (x_n) were almost convergent to 0, then by definition, for every $m \in \mathbb{N}$ there would exist N_c such that for all $N \geq N_c$,

$$\frac{1}{N} \sum_{k=1}^N d(x_{k+m}, 0) \ll c.$$

In particular, this must hold for $m = 0$. That is,

$$\frac{1}{N} \sum_{k=1}^N d(x_k, 0) \ll 1 \quad \text{for all sufficiently large } N.$$

However, taking $N = 2^J$ we obtain

$$\frac{1}{2^J} \sum_{k=1}^{2^J} d(x_k, 0)(1) \geq \frac{1}{2^J} \sum_{j=1}^J (2^j + 4^j) \geq \frac{1}{2^J} \cdot \frac{4^{J+1} - 4}{3} = \frac{4}{3} \cdot 2^J - \frac{4}{3 \cdot 2^J} \rightarrow \infty,$$

which contradicts $\frac{1}{N} \sum_{k=1}^N d(x_k, 0) \ll 1$. Therefore, (x_n) is not almost convergent.

A more detailed verification can be carried out using the scalarization $\rho = \xi_e \circ d$.

4 Almost Statistical Sequential Compactness

Definition 5 A subset A of a cone metric space (X, d) is called almost statistically sequentially compact if every sequence in A has a subsequence which is almost statistically convergent to some point in A .

Definition 6 A subset A of a cone metric space (X, d) is called almost sequentially compact if every sequence in A has a subsequence which is almost convergent to some point in A .

Lemma 5 (Subsequence inheritance for bounded shifts) Let $x_n \xrightarrow{asc} a$. Let (n_k) be an increasing sequence with $\lim_{k \rightarrow \infty} n_k/k = 1$. Then for every $c \in \text{int}P$ and every fixed $M_0 \in \mathbb{N}$,

$$\lim_{N \rightarrow \infty} \sup_{0 \leq m \leq M_0} \frac{1}{N} |\{k \leq N: d(x_{n_k+m}, a) \ll c\}| = 0.$$

Proof. Fix $c \in \text{int}P$ and $M_0 \in \mathbb{N}$. For each fixed $m \in \{0, 1, \dots, M_0\}$, the shifted sequence (x_{n_k+m}) is also asc to a by definition. Hence, for every $\varepsilon > 0$ there exists N_0 such that for all $M \geq N_0$,

$$\sup_{r \in \mathbb{N}} \frac{1}{M} |\{j \leq M: d(x_{j+r+m}, a) \ll c\}| < \varepsilon.$$

Now fix $r \in \mathbb{N}$ and set $M = n_N$. Since $n_N/N \rightarrow 1$, we have $M/N \rightarrow 1$ and

$$\frac{1}{N} |\{k \leq N: d(x_{n_k+r+m}, a) \ll c\}| \leq \frac{1}{N} |\{j \leq n_N: d(x_{j+r+m}, a) \ll c\}|$$

$$= \frac{n_N}{N} \cdot \frac{1}{n_N} |\{j \leq n_N: d(x_{j+r+m}, a) \ll c\}|.$$

Take $\sup_r$ and then $N \rightarrow \infty$ to get the claim for each fixed m . Finally take $\sup_{0 \leq m \leq M_0}$ (finite supremum).

Theorem 4 Every almost sequentially compact subset of X is almost statistically sequentially compact.

Proof. Let A be almost sequentially compact and $(x_n) \subset A$. Then there exists a subsequence (x_{n_k}) almost convergent to some $a \in A$. By Theorem 3, this subsequence is almost statistically convergent to a .

Theorem 5 Every sequentially compact subset of X is almost statistically sequentially compact.

Proof. Let A be sequentially compact and $(x_n) \subset A$. Then there exists a subsequence (x_{n_k}) convergent to $a \in A$. Convergence implies almost convergence, which in turn implies almost statistical convergence.

Theorem 6 Every almost statistically sequentially compact subset of X is bounded.

Proof. Suppose A is unbounded. Fix $a \in A$ and $c_0 \in \text{int}P$. Choose $x_n \in A$ such that $d(x_n, a) \ll nc_0$. For any subsequence (x_{n_k}) , the set of indices where $d(x_{n_k}, a) \ll c_0$ has positive uniform density, so no subsequence can be almost statistically convergent to any point. Indeed, for any subsequence (x_{n_k}) we have $d(x_{n_k}, a) \ll kc_0$ for all k , hence in particular $d(x_{n_k}, a) \ll c_0$ for all $k \geq 1$. Thus for every N and every shift m ,

$$|\{k \leq N: d(x_{n_k+m}, a) \ll c_0\}| = N,$$

so the defining $\sup_m$ -density cannot tend to 0.

Proposition 1 Every almost statistically sequentially compact subset of X is closed.

Proof. Let $(x_n) \subset A$ converge to x . By almost statistical sequential compactness, there exists a subsequence (x_{n_k}) almost statistically convergent to some $a \in A$. Since convergence implies almost statistical convergence, the whole sequence is almost statistically convergent to x . By uniqueness, $x = a \in A$, so A is closed.

Example 3 $A = \{0\} \cup \{2^j : j \in \mathbb{N}\} \subset \mathbb{R}$ with the metric $d(x, y) = (|x - y|, |x - y|^2)$ is almost statistically sequentially compact but not sequentially compact.

5 Almost Statistical Completeness

Definition 7 Let (X, d) be a cone metric space. A sequence (x_n) in X is called almost statistically Cauchy if for every $c \in \text{int}P$ there exists $N \in \mathbb{N}$ such that

$$\limsup_{M \rightarrow \infty} \frac{1}{M} |\{k \leq M : d(x_{m+k}, x_N) \ll c\}| = 0.$$

Definition 8 A cone metric space (X, d) is said to be almost statistically complete if every almost statistically Cauchy sequence has a subsequence which converges in the usual cone metric sense.

Theorem 7 Let (x_n) be almost statistically convergent to $a \in X$. Then,

- (i) There exists a subsequence (x_{n_k}) such that $x_{n_k} \rightarrow a$.
- (ii) For every $c \in \text{int}P$, the set

$$\{n \in \mathbb{N} : d(x_n, a) \ll c\}$$

has natural density one.

Proof. Assume $x_n \xrightarrow{asc} a$. Let $c \in \text{int}P$. By definition,

$$\limsup_{N \rightarrow \infty} \frac{1}{N} |\{k \leq N : d(x_{k+m}, a) \ll c\}| = 0.$$

In particular, for $m = 0$,

$$\lim_{N \rightarrow \infty} \frac{1}{N} |\{k \leq N : d(x_k, a) \ll c\}| = 0.$$

Thus the set

$$A_c := \{n \in \mathbb{N} : d(x_n, a) \ll c\}$$

has natural density one, hence is infinite.

Choose a sequence $(c_j) \subset \text{int}P$ such that $c_{j+1} \ll c_j$ and $c_j \rightarrow 0$. Since each A_{c_j} is infinite, we may inductively choose an increasing sequence (n_j) such that $n_j \in A_{c_j}$.

Then,

$$d(x_{n_j}, a) \ll c_j \rightarrow 0,$$

which implies $x_{n_j} \rightarrow a$.

Theorem 8 Let (X, d) be a complete cone metric space. Then every almost statistically Cauchy sequence has a subsequence which converges in the usual cone metric sense. Consequently, every almost statistically Cauchy sequence has at least one ordinary limit point in X .

Proof. Let (x_n) be almost statistically Cauchy. Choose a sequence $(c_j) \subset \text{int}P$ such that

$$c_{j+1} \ll c_j \quad (j \in \mathbb{N})$$

and

$$c_j \rightarrow 0$$

in the cone metric sense.

For each $j \in \mathbb{N}$, since (x_n) is almost statistically Cauchy, there exists $N_j \in \mathbb{N}$ such that

$$\limsup_{M \rightarrow \infty} \sup_{m \in \mathbb{N}} \frac{1}{M} |\{k \leq M: d(x_{m+k}, x_{N_j}) \ll c_j\}| = 0.$$

We claim that the subsequence (x_{N_j}) is Cauchy. Let $c \in \text{int}P$ be arbitrary. Since $c_j \rightarrow 0$, there exists $j_0 \in \mathbb{N}$ such that

$$c_j + c_j \ll c \quad \text{for all } j \geq j_0.$$

Fix $i, j \geq j_0$ with $j < i$. By the defining property of N_j and N_i , there exists $M_0 \in \mathbb{N}$ such that for all $M \geq M_0$,

$$\frac{1}{M} |\{k \leq M: d(x_{N_j+k}, x_{N_j}) \ll c_j\}| < \frac{1}{2}$$

and

$$\frac{1}{M} |\{k \leq M: d(x_{N_j+k}, x_{N_i}) \ll c_i\}| < \frac{1}{2}.$$

Define

$$A_M := \{k \leq M: d(x_{N_j+k}, x_{N_j}) \ll c_j\}, \quad B_M := \{k \leq M: d(x_{N_j+k}, x_{N_i}) \ll c_i\}.$$

Then for all $M \geq M_0$,

$$|A_M| > \frac{M}{2}, \quad |B_M| > \frac{M}{2},$$

hence

$$A_M \cap B_M \neq \emptyset.$$

Choose $k \in A_M \cap B_M$. Then

$$d(x_{N_j+k}, x_{N_j}) \ll c_j \quad \text{and} \quad d(x_{N_j+k}, x_{N_i}) \ll c_i.$$

By symmetry,

$$d(x_{N_j}, x_{N_j+k}) = d(x_{N_j+k}, x_{N_j}) \ll c_j.$$

Since $d(x_{N_j}, x_{N_j+k}) \ll c_j$, there exists $u_j \in \text{int}P$ such that

$$d(x_{N_j}, x_{N_j+k}) + u_j = c_j.$$

Similarly, since $d(x_{N_j+k}, x_{N_i}) \ll c_i$, there exists $u_i \in \text{int}P$ such that

$$d(x_{N_j+k}, x_{N_i}) + u_i = c_i.$$

Using the triangle inequality, we obtain

$$d(x_{N_j}, x_{N_i}) \leq d(x_{N_j}, x_{N_j+k}) + d(x_{N_j+k}, x_{N_i}) \leq c_j + c_i - (u_j + u_i).$$

Since $u_j + u_i \in \text{int}P$, it follows that

$$d(x_{N_j}, x_{N_i}) \ll c_j + c_i \ll c.$$

Thus, for every $c \in \text{int}P$, there exists j_0 such that

$$d(x_{N_j}, x_{N_i}) \ll c \quad \text{whenever } i, j \geq j_0.$$

Hence (x_{N_j}) is a Cauchy sequence in the usual cone metric sense.

Since (X, d) is complete, there exists $a \in X$ such that

$$x_{N_j} \rightarrow a.$$

Therefore, (x_n) has a subsequence converging to a . In particular, a is an ordinary limit point of the sequence (x_n) .

6 Conclusion

In this paper, we have introduced and systematically investigated the notions of almost convergence and almost statistical convergence in cone metric spaces.

By integrating Lorentz-type almost convergence with uniform density techniques, we established a framework that extends classical and statistical convergence to a genuinely cone-ordered setting. We first developed the fundamental properties of almost convergence and almost statistical convergence and clarified their relationship. It was shown that almost convergence implies almost statistical convergence, while the converse fails in general. This separation demonstrates that almost statistical convergence forms a strictly weaker yet structurally meaningful mode of convergence in cone metric spaces.

We then introduced the concepts of almost statistical sequential compactness and almost statistical completeness. A hierarchy between classical compactness, almost compactness, and almost statistical compactness was established. In particular, we proved that sequential compactness and almost compactness both imply almost statistical compactness, while the converses do not hold in general.

Furthermore, we formulated the notion of almost statistically Cauchy sequences and showed that in complete cone metric spaces, every almost statistically Cauchy sequence admits a convergent subsequence. Thus, almost statistical completeness appears as a natural weakening of classical completeness that remains compatible with the cone structure.

The results obtained in this work open several directions for further research. In particular, the interaction between almost statistical convergence and additional geometric assumptions on the cone (such as normality or regularity), as well as extensions to operator sequences, double sequences, and vector-valued summability methods, deserve further investigation. These shift-uniform convergence notions are expected to be useful in cone-valued fixed point theory and dynamical systems, for instance in formulating stability and convergence of iterates under weaker (density-based) asymptotic regimes.

We believe that almost statistical methods provide a flexible and robust analytical tool for studying generalized convergence phenomena in ordered metric structures.

It should be noted that the notion of almost statistical completeness introduced in this paper is weaker than classical completeness, as it guarantees the existence of a convergent subsequence rather than full convergence. Strengthening this notion remains an open problem.

Conflict of Interest. The author declares that he has no competing interests.

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