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Conference paper

SOME ALMOST INTEGERS GENERATED BY SOME PISOT NUMBERS

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Abstract

Inspired by the Euler identity, there have been a lot of attempts to approximate particular integers by applying some algebraic operations to some non-integer numbers. Here it is used a method of generating some almost integers with the help of Pisot numbers (real algebraic integers greater than one with all Galois conjugates located in the open unit disc in the complex plane). The method consists of obtaining values that are very close to whole numbers by taking high powers of Pisot numbers. There are used some terms of three polynomial sequences to generate some Pisot numbers. A desired approximation to the integer depending on the power of the Pisot number and the distance of its Galois conjugates from the origin will be examined.

Keywords: *Almost integers, Pisot numbers, Algebraic integers, Galois conjugate.*

1. Introduction

By an almost integer, we define a very close value (but not equal) to an integer. Such examples are:

$$e^{\pi} - \pi = 19,999099979... \approx 20$$
$$\frac{\pi^9}{e^8} = 9,9998387... \approx 10$$
$$\sin 11 = -0.9999999922... \approx -1$$
$$\left(\frac{23}{9}\right)^5 = 109,000338... \approx 109$$
$$88 \ln 89 = 395,000000536... \approx 395 \text{ etc.}$$

The inspiration for searching such values comes from the famous Euler's identity $e^{i\pi} + 1 = 0$ i.e. $e^{i\pi} = -1$. In most cases, almost integers, as values of some expressions, occur by a coincidence, but there are a case when there is given a method for generating such values. Such a method is given in [4].

2. Pisot Numbers

Before defining Pisot numbers, let us recall some of the definitions needed for that purpose.

Let $P_n(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$ be a polynomial with integer coefficients.

Definition 2.1: The polynomial $P_n(x)$ is said to be <u>*monic*</u> if $a_n = 1$.

Definition 2.2: The polynomial $P_n(x)$ is said to be <u>*irreducible over integers*</u> if it cannot be factored into a product of two non-constant polynomials with integer coefficients.

Example 2.1: $Q(x) = x^3 + 1$ is not irreducible over integers (or it is reducible) since $Q(x) = (x+1)(x^2 - x + 1)$ but $R(x) = x^2 + 1$ is irreducible.

Definition 2.3: Let α be a complex number. The monic polynomial of least degree with integer coefficients, having α as a root is called *the minimal polynomial of* α.

Definition 2.4: A complex number is called *an algebraic integer* if it is a root of a polynomial with integer coefficients.

Let α be an algebraic integer and $P(x)$ its minimal polynomial.

Definition 2.5: All the roots $\theta \neq \alpha$ of the polynomial $P(x)$ are caled *Galois conjugates of* α.

We are now ready to define Pisot numbers:

Definition 2.6: A real algebraic integer larger than 1 whose all Galois conjugates lie on the interior of the complex unite disc is called *a Pisot number*.

These numbers were first studied by Thue [7] in 1912 and Hardy [1] in 1919, but they gained popularity by Pisot [4] in 1938.

More about Pisot numbers can be found in [2, 4-6, 8].

3. Almost Integers Generated by Pisot Numbers

In this section, we will show that higher powers of Pisot numbers tend to go closer to integers. Before that, let us define what do we mean by "close to an integer".

Definition 3.1: Let $\{x\}$ denoted the decimal part of the real number x. We define $\|x\| = \begin{cases} \{x\} \text{ if } \{x\} \leq 0,5 \\ 1 - \{x\} \text{ if } \{x\} > 0,5 \end{cases}$ to be the distance from x to the nearest integer.

Theorem 3.1: [3] If α is a Pisot number, then $\lim_{n\to\infty} \|\alpha^n\| = 0$.

To prove this theorem, we will use the following result:

Theorem 3.2: [3] Let $P(x)$ be a monic, irreducible polynomial of degree d with (not necessarily distinct) roots $\theta_1, \theta_2, ..., \theta_d$. Then $\theta_1^n + \theta_2^n + ... + \theta_d^n$ is an integer, for all positive integers n.

Proof of theorem 3.1: [3] Let $\alpha = \theta_1$ have the Galois conjugates $\theta_2, \theta_3, ..., \theta_d$. Since $|\theta_i| < 1,\ \forall i = 2,3,...,d,$ then $\lim_{n\to\infty} \theta_i^n = 0,\ \forall i = 2,3,...,d.$ So $\lim_{n\to\infty} \sum_{i=2}^{d} \theta_i^n = 0 ...(1).$

From the theorem 3.2 we have that $\alpha^n + \theta_2^n + \theta_3^n + ... + \theta_d^n = b_n \in □$, for all $n \in □$. So $b_n - \alpha^n = \sum_{i=2}^{d} \theta_i^n$ where $b_n \in □$, for all $n \in □$. Then, from the last equation, we get $\lim_{n\to\infty}\left(b_n - \alpha^n\right) = \lim_{n\to\infty} \sum_{i=2}^{d} \theta_i^n ...(2).$

Replacing (1) in (2) we get $\lim_{n\to\infty}\left(b_n - \alpha^n\right) = 0,$ and since b_n is an integer, it implies that $\lim_{n\to\infty} \|\alpha^n\| = 0.$

In [6] there are given three polynomial sequences in which positive zeros are Pisot numbers. We will use these polynomials to generate some Pisot numbers with the help of which we will construct some almost integers.

$$P_n(x) = x^n(x^2 - x - 1) + x^2 - 1,\ n = 1,2,3,...$$

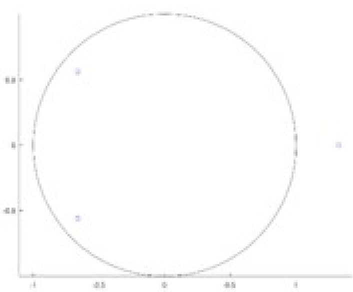
$$Q_n(x) = x^n - \frac{x^{n+1} - 1}{x^2 - 1},\ n = 3,5,7,...$$

$$R_n(x) = x^n - \frac{x^{n-1} - 1}{x - 1},\ n = 3,5,7,...$$

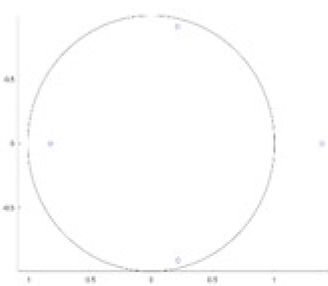
Next, we represent graphically some Pisot numbers generated by these polynomial sequences as well as its corresponding Galois conjugates:

3. Table Figures and Equations

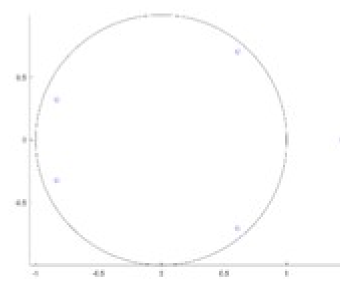
$P_1(x) = x^3 - x - 1$



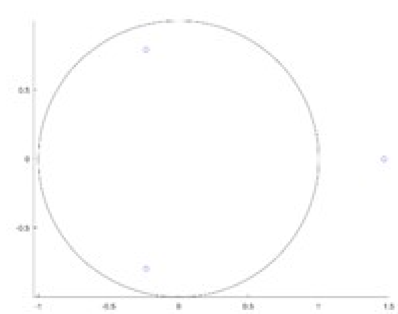
$P_2(x) = x^4 - x^3 - 1$



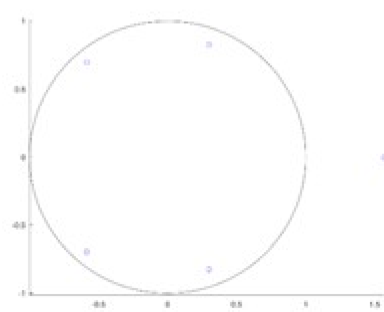
$P_3(x) = x^5 - x^4 - x^3 + x^2 - 1$



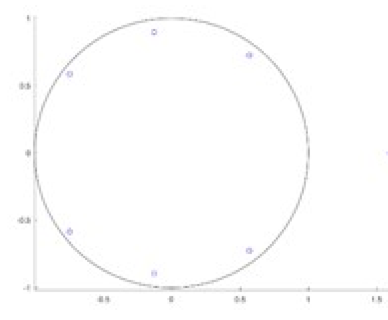
$Q_3(x) = x^3 - x^2 - 1$



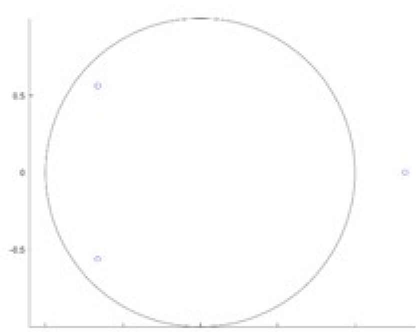
$Q_5(x) = x^5 - x^4 - x^2 - 1$



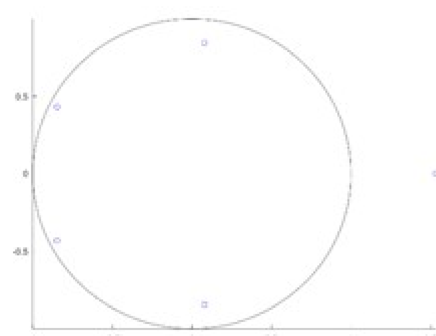
$Q_7(x) = x^7 - x^6 - x^4 - x^2 - 1$



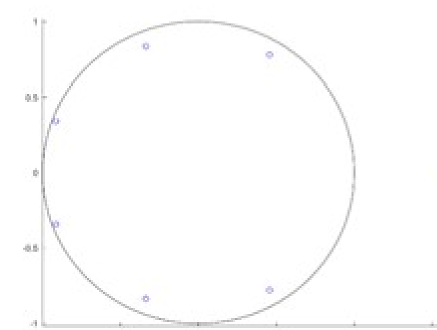
$P_1(x) = x^3 - x - 1$



$R_5(x) = x^5 - x^3 - x^2 - x - 1$



$R_7(x) = x^7 - x^5 - x^4 - x^3 - x^2 - x - 1$



In the next tables we give values of some powers of some Pisot numbers generated by these polynomial sequences:

Table 1. Some powers of some Pisot numbers generated by these polynomial sequences

	P_1		P_2		P_3		φ	
n	α^n	∑θ_i	α^n	∑θ_i	α^n	∑θ_i	α^n	θ
	1.32472		1.38028		1.44327		1.61803	0.61803
1	1.32472	1.51	1.38028	2.59	1.44327	3.33	1.61803	0.61803
2	1.75488	1.14	1.90517	2.24	2.08303	2.78	2.6180211	0.3819611
3	2.32472	0.86	2.62967	1.69	3.00637	2.33	4.2360366	0.0557259
4	3.07960	0.65	3.62968	1.23	4.33901	1.95	6.8540344	9.643E-06
5	4.07960	0.49	5.00998	1.08	6.26236	1.64	11.090033	8.34E-26

Table 2. Some powers of some Pisot numbers generated by these polynomial sequences

	Q_3		Q_5		Q_7		φ	
n	α^n	∑θ_i	α^n	∑θ_i	α^n	∑θ_i	α^n	θ
	1.46557		1.57015		1.60135		1.61803	0.61803
1	1.46557	1.36	1.57015	3.19	1.60135	5.13	1.61803	0.61803
2	2.14790	0.93	2.46537	2.55	2.56432	4.4	2.618021	0.381961
3	3.14789	0.64	3.87100	2.04	4.10638	3.77	4.236037	0.055726
4	4.61345	0.43	6.07805	1.64	6.57575	3.24	6.854034	9.64E-06
5	6.76134	0.3	9.54346	1.31	10.53007	2.79	11.09003	8.34E-26

Table 3. Some powers of some Pisot numbers generated by these polynomial sequences

	R_3		R_5		R_7		φ	
n	α^n	∑θ_i	α^n	∑θ_i	α^n	∑θ_i	α^n	θ
	1.32472		1.38028		1.44327		1.61803	0.61803
1	1.32472	1.51	1.38028	1.77	1.44327	5.13	1.61803	0.61803
2	1.75488	1.12	1.90517	1.56	2.08303	4.41	2.618021	0.381961
3	2.32472	0.86	2.62967	1.38	3.00637	3.81	4.236037	0.055726
4	3.07960	0.65	3.62968	1.22	4.33901	3.31	6.854034	9.64E-06
5	4.07960	0.49	5.00998	1.08	6.26236	2.89	11.09003	8.34E-26

4. Integer Approximations

On the previous tables, we added the value of φ (as a Pisot number generated by the polynomial $P(x) = x^2 - x - 1$) to compare the results with the previous Pisot numbers.

We can see that powers of φ tend to go more quickly to the integer than our other Pisot numbers.

This happens because Galois conjugates of φ is closer to the origin while Galois conjugates of our other Pisot numbers are closer to the unit circle.

From the proof of theorem 3.1 we can notice that the "closeness" of a power of a Pisot number to an integer depends on its Galois conjugates. Namely $\lim_{n\to\infty}\left(b_n-\alpha^n\right)=\lim_{n\to\infty}\sum_{i=2}^{d}\theta_i^n$ implies

$$\lim_{n\to\infty}\left|b_n-\alpha^n\right|=\lim_{n\to\infty}\left|\sum_{i=2}^{d}\theta_i^n\right|\le\lim_{n\to\infty}\sum_{i=2}^{d}\left|\theta_i\right|^n\le\sum_{i=2}^{d}\left|\theta_i\right|.$$

On the previous tables we have calculated $\sum_{i=2}^{d}\left|\theta_i\right|$ for our cases.

5. Conclusion

The approximation of integers makes almost integers an interesting object of study. Theorem 3.1 can be used to generate such numbers by using Pisot numbers.

However, by some examples, we showed that some Pisot numbers tend to approximate integers more quickly than some others and we also showed that this depends on the distance of its Galois conjugates from the origine.

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