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GREEN'S RELATIONS IN SEMIGROUPS

Merita Azemi[1], Rushadije Halili[1]

[1*]*Department of Mathematics, Faculty of Natural Science and Mathematics, University of Tetova, RNM*

[*]*Corresponding author e-mail: merita.azemi@unite.edu.mk*

Abstract

In this paper, we discuss a very useful tool in the study of monoids/semigroups called Green's relations. At this point, we note that every $\mathbf{J}$ - class decomposes into a set of $\mathbf{R}$ - classes as well as into a set of $\mathbf{L}$- classes. These facts can be represented by a schema called an "egg-box". We say that a $\mathbf{D}$ - class (or a $\mathbf{H}$ - class or $\mathbf{R}$- class or $\mathbf{L}$- class) is regular if it contains an idempotent and that any two maximal subgroups contained in the same $\mathbf{D}$ - class of a monoid M are isomorphic. If R is a minimal right ideal and L a minimal left ideal, then $R \cap L$ is a maximal subgroup of S . Also we say that the maximal subgroup of a semigroup S coincide with the $\mathbf{H}$ - classes of S which contains idempotents. These facts can be represented by a schema called an "egg-box".

Keywords:*Semigroup, monoid, classes, ideals, idempotent.*

1. Introduction

In the first and second section, we refer to [1], [3], [4] reference. Throughout third section *we* shall call upon results on Green' relations with egg – box schema in *semigroups* and *we refered to the [2], [5], [6] reference*.

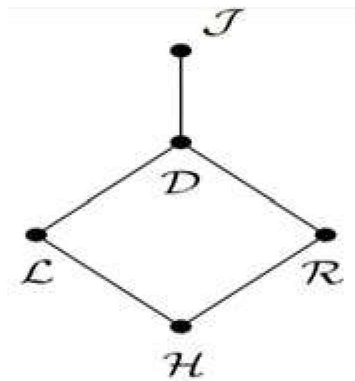
Definition 1.1 $a\,\mathbf{L}, b \Leftrightarrow S^1a = S^1b$, $\forall a,b \in S$ $\Leftrightarrow \exists s,t \in S^1$ with $a = sb$ and $b = ta$.

Definition 1.2 $a\,\mathbf{R}\,b \Leftrightarrow aS^1 = bS^1 \Leftrightarrow \exists s,t \in S^1$ with $a = bs$ and $b = at$.

Definition 1.3 $a\ \mathbf{H}\ b \Leftrightarrow a\ \mathbf{L}, b \wedge a\ \mathbf{R}\ b$, $\mathbf{H} = \mathbf{L}, \cap \mathbf{R}.$

Definition 1.4 $a\ \mathbf{J}\ b \Leftrightarrow S^1aS^1 = S^1bS^1 \Leftrightarrow \exists s,t,u,v \in S^1$ with $a = sbt$ and $b = uav$.

Definition 1.5 $a\,\mathbf{D}\,b \Leftrightarrow \exists c \in S\ \ a\,\mathbf{R}\,c\,\mathbf{L}\ b \Leftrightarrow \exists c \in S, a\ \mathbf{L}\,, c$ and $c\,\mathbf{R}\,b.$



2. Connection of Green's Classes

Lema 2.1 $\mathbf{L} \circ \mathbf{R} = \mathbf{R} \circ \mathbf{L}$

Proof: Let $m\, \mathbf{RL}\, n$ where $m, n \in S$. There exist $p \in M$ such that $m\, \mathbf{R}\, p$, $p\, \mathbf{L}\, n$ and by the definitions $u, u', v, v' \in S$: $p = mu,\ m = pu',\ n = vp,\ p = v'n$. Let $q = vm$, we than have $q = vm = v(pu') = (vp)u' = nu'$, $m = vp = v(mu) = (vm)u = qu$. This shows that $q\, \mathbf{R}\, n$.

Also we have $m = pu' = (v'n)n' = v'(nu') = v'q$. Since $q = vm$ by the definition of q, we obtain $m\, \mathbf{L}\, q$. Therefore $m\, \mathbf{L}\, q\, \mathbf{R}\, n$ and $m\, \mathbf{L}\, \mathbf{R} n$. This proves the inclusion RL $\subset$ $\mathbf{R}\, \mathbf{L}$. The proof of converse inclusion is symmetrical.

Hence the binary relations defined by $a\, \mathbf{D}\, b \Leftrightarrow a\, \mathbf{L}\, x\, \mathbf{R}\, b$ for some $x \in S$

$\Leftrightarrow a\, \mathbf{R}\, y\, \mathbf{L}\, b$ for some $y \in S$ is equivalence relation.

Corollary 2.1 Let $e \in E(S)$ and $e \in R_e$. Then we have

$$a\, \mathbf{R}\, e \Rightarrow ea = a,$$

$$a\, \mathbf{L}\, e \Rightarrow ae = a,$$

$$a\, \mathbf{H}\, e \Rightarrow a = ae = ea.$$

Proof: Let G be subgroup with idempotent e. Then for any $a \in G$ we have $ea = a = ae$ and there exist $a^{-1} \in G$ with $aa^{-1} = e = a^{-1}a$. Then

$$\begin{cases} ea = a \\ aa^{-1} = e \end{cases} \Rightarrow a\, \mathbf{R}\, e \quad \text{and} \quad \begin{cases} ae = a \\ a^{-1}a = e \end{cases} \Rightarrow a\, \mathbf{L}, e \quad \Rightarrow a\, \mathbf{H}\, e$$
. Therefore $G \subseteq H_e$.

This in turn implies the well known fact that there is at most one idempotent in each $\mathbf{H}$ - class, for if $e^2 = eHf = f^2$ then $e = ef = f$.

Lemma 2.2 (Green's Lemma) Assume $a,b \in S$ such that $a \, \mathbf{R} \, b$ and let $s,s' \in S$ such that

$$as = b \quad \text{and} \quad bs' = a.$$

Then $\rho_S : L_a \to L_b$ are mutually inverse, $\mathbf{R}$ - class preserving bijections. So if $c \in L_a$, then $c \, \mathbf{R} \, cp_s$ and if $s \in L_b$ then $d \, \mathbf{R} \, dp_s$.

Proof: If $c \in L_a$ then $cp_s = cs \, \mathbf{L}, as = b$ because $\mathbf{L},$ is a right congruence. So $cp_s \, \mathbf{L} \, b$ therefore $\rho_s : L_a \to L_b$. Dually $\rho_{s'} : L_b \to L_a$.

Lemma 2.3 (Continuing Green's Lemma) For any $c \in L_a$ we have $\rho_s : H_c \to H_{cs}$ is a bijection with inverse $\rho_{s'} : H_{cs} \to H_c$. In particular, put $c = a$ then $\rho_s : H_a \to H_b$ and $\rho_s : H_b \to H_a$ are mutually inverse bijections. For any $s \in S^1$, $\lambda_s : S \to S$ is given by $a\lambda_s = sa$.

Corollary 2.2 If $a \, \mathbf{D} \, b$ then there exist a bijection $H_a \to H_b$.

Proof: If then $a \, \mathbf{D} \, b$ then there exist $h \in S$ with $a \, \mathbf{R} \, h \, \mathbf{H} \, b$. Then exist a bijection $H_a \to H_h$ by Green's Lemma and we also have that there exist a bijection $H_a \to H_h$ by dual of Green's Lemma. Therefore there exist a bijection $H_a \to H_b$. Thus any two $\mathbf{H}$ - classes in the same $\mathbf{D}$ – class have the same cardinality.

3. Main results

Definition 3.1 Idempotent of a semigroup S is an element a such that $a^2 = a$. Also we define the set of idempotents in S to be $E(S) = \{e \in S / e^2 = e\}$.

Definiton 3.2 $a \in S$ is regular if $a = axa$ for some $x \in S$. a' is an inverse of a if $a = aa'a \wedge a'aa'$.

Proposition 3.1 In a semigroup $ab \in R_a \cap L_b$ if and only if $R_b \cap L_a$ contains an idempotent.

Proof: First note that in all cases $ab \le a$ ($\mathbf{R}$) and $ab \le b$($\mathbf{L},$). Asume $ab \in R_a \cap R_b$, in particular $a \, \mathbf{D} \, b$ and a, b , ab are located as follows in the egg-box picture $D_a = D_b$;

a			ab	
e			b	

Then $rab = b$ for some $r \in S^1$ and it follows from Green's Lemma that: $\bar{a}: R_b \to R_{ab}, x \mapsto ax$ and $\bar{r}: R_{ab} \to R_b, y \mapsto ry$ are mutually inverse, **L,** - class preserving bijections. Since $a \in R_{ab}$ this implies $ra \in R_b \cap L_a$. Also $rax = x, \forall x \in R_b$, in particular $e = ra$ is idempotent. Conversely assume that $R_b \cap R_a$ contains an idempotent e. Then $a \in S^1 e$ and $ae = a$. By Green's Lemma
$\bar{a}: R_e \to R_a, x \mapsto ax$ is an **L,** - class preserving bijection.Since $b \in R_e$ this implies $ab \in R_a \cap R_b$. □

Proposition 3.2 Let S be any semigroup and $a, b \in S$. Then the **H** - class H_b contains an inverse a' of a if and only if there are idempotents $e \in R_a \cap L_b$ and $f \in R_b \cap L_a$, in which case $e = aa'$, $f = a'a$ and a' is the only inverse of a in H_b .

Proof: Suppose there is an inverse a' for a in the **H** - class H_b. Then we have the egg-box:

a	$\cdots$	aa'
$\vdots$		$\vdots$
$a'a$	$\cdots$	b, a'

So there are idempotents $aa' \in R_a \cap L_b$ and $aa' \in R_b \cap L_a$. In this case a' is the only inverse of a in H_b, for if a'' is another then by uniqueness of idempotents in **H** - classes we have the egg-box diagram.

a	$\cdots$	$aa' = aa''$
$\vdots$		$\vdots$
$a'a = a''a$	$\cdots$	b, a', a''

And so $a' = a'aa' = a''aa' = a''aa'' = a''$.Conversely, suppose there are idempotents $e \in R_a \cap L_b$ and $f \in R_b \cap L_a$. Then by part (ii) of Green's Lemma since $a = af$ **L,** f there is an element $a' \in R_f \cap L_e = H_b$ such that the egg-box diagram

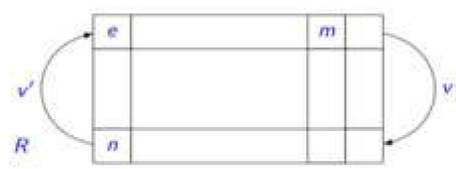
a	$\cdots$	$e = aa'$
$\vdots$		$\vdots$
f	$\cdots$	b, a'

holds and we have $aa'a = ea = a$ and $a'aa' = a'e = a'$ so a' is an inverse of a. By uniqueness of idempotents in **H** - classes we have $f = aa'$ and we are done.

Proposition 3.3 Let S be a monoid and let D be a **D** - class of S. The following conditions are equivalent:

(i) D - contains an idempotent.
(ii) Each **R** - class of D contains an idempotent.
(iii) Each **L,** - class of D contains an idempotent.

Proof: We know that (i) implies (iii) and (ii) implies (iii). Let $e \in D$ be an idempotent. Let R be an **R** - class of D. The **H** - class $H = L(e) \cap R$ is nonempty. Let n be an element of H. Since n **L** e there exist $v, v' \in M$ such that $n = ve$, $e = v'n$. Let $m = ve'$. Then $mn = e$ because $mn = (ev')n = e(v'n) = ee = e$. Moreover, we have m **R** e since $mn = e$ and $m = ev'$. Therefore, $e = mn$ is in $R(m) \cap L(n)$. This implies, by last proposition that $R = R(n)$ contains an idempotent.



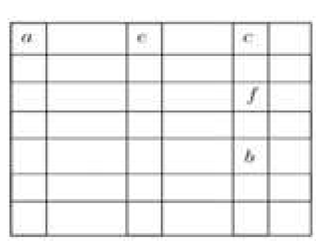
A. **D** - class satisfying one of the conditions of the last proposition is called regular.

Lemma 3.1

(i) If $a = axa$ then $ax, xa \in E(S)$ and a **R** xa,
(ii) If b **R** $f \in E(S)$ then b is regular,
(iii) If b **L,** $f \in E(S)$, then b is regular.

Proof: We know that (i) is always the truth. (ii) If b **R** f then $fb = b$. Also $f = bs$ for some $s \in S^1$. Therefore $b = fb = bsb$ and so b is regular. (iii) is dual to (ii).

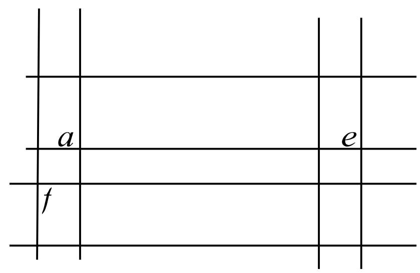
Lemma 3.2 (Regular **D** - class Lemma). If a **D** b then if a is regular, so is b.
Proof. Let a be regular with a **D** b. Then a **R** c **L** ,b for some $c \in S$.



There exists $e=e^2$ with $e\,\mathbf{R}\,a\,\mathbf{R}\,c$ by (ii) above. By (ii) c is regular. By (i), $c\,\mathbf{L}\,f=f^2$. By (iii) b is regular.

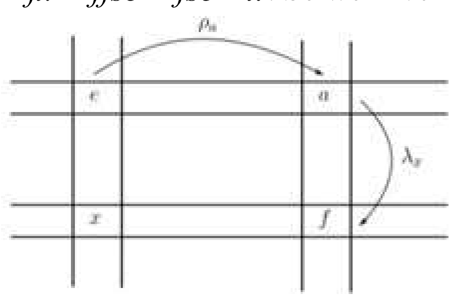
Corollary 3.1 $e\,\mathbf{D}\,f \Rightarrow H_e \cong H_f$.

Proof: Suppose $e,f\in E(S)$ and $e\,\mathbf{D}\,f$. There exists $a\in S$ with $e\,\mathbf{R}\,a\,\mathbf{L}\,f$.



As $e\,\mathbf{R}\,a$ there exists $s\in S^1$ with $e=as$ and $ea=a$. So $a=asa$. Put $x=fse$ then $ax=afse=ase=e^2=e$ and so $a=ea=axa$. Since a **L,** f there exists $t\in S^1$ with $ta=f$. Then $xa=fsea=fsa=tasa=ta=f$.

Also $xax=fx=ffse=fse=x$. So we have the diagram



$e=axa=axax=xaxf=xa$. We have $ea=a$ therefore $\rho_a: H_e\to H_a$ is bijection.

From a **L,** f and $xa=f$ we have $\lambda_x: H_a\to H_f$ is bijection. Hence $\rho_a\lambda_x: H_e\to H_f$ is a bijection. Let $h,j\in H_e$ then $h(\rho_a\lambda_x)k(\rho_a\lambda_x)=(xha)(xka)=xh(ax)ka=xheka=xhka=hk(\rho_a\lambda_x)$. So $\rho_a\lambda_x$ is an isomorphism and $H_e\cong H_f$.

Conclusions

This paper focused in the study of Green's relations through egg-box schema. Using the facts that **L,** , **R** $\subseteq$ **D** $\subseteq$ **J** , that there is at most one idempotent in each **H** - class, for if $e^2=eHf=f^2$ then $e=ef=f$ and that any two **H** - classes in the same **D** - class have the same cardinality, we show that if $a\,\mathbf{D}\,b$ then if a is regular, so is b. Also with the help of egg - box schema we conclude that $H_e\cong H_f$.

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