

RELATION BETWEEN MULTIALGEBRAS AND BOOLEAN ALGEBRAS WITH OPERATOR

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Abstract

In this paper, we will show that based on the concept of multialgebras and relying on the power structures of these algebras, which we could also consider as relational structures, and adding the set of operations $\cup, \cap, -$ we get a Boolean algebra with operator. The power structure of relational structure $\square = (A, \square)$ is an algebra with the set $\square(A)$ and the set of all basic operations defined $\{R^* \mid R \in R\}$ as follows: for $R \in R_{n+1}$ we have

$R^* : P(A)^n \rightarrow P(A)$ so that for $X_0, \dots, X_{n-1} \in P(A)$,

$R^*(X_0, \dots, X_{n-1}) = \{y \in A \mid (\forall i \in \{0, \dots, n-1\})(\exists x_i \in X_i)(x_0, \dots, x_{n-1}, y) \in R\}$. Hence if B is Boolean algebra, for every f we say that is an operator if it is additive for every its argument, thus:

$f(x_0, \dots, \Sigma\{y_i \mid i \in \{0, \dots, k-1\}\}, \dots, x_{n-1}) = \Sigma\{f(x_0, \dots, y_i, \dots, x_{n-1}) \mid i \in \{0, \dots, k-1\}\}$.

Furthermore if $f(x_0, \dots, \Sigma\{y_i \mid i \in I\}, \dots, x_{n-1}) = \Sigma\{f(x_0, \dots, y_i, \dots, x_{n-1}) \mid i \in I\}$ than f is complete additive operator. Also we will show that every Boolean algebra with operator is a power structure of any multialgebra.

Keywords: Multialgebra, Boolean algebra with operator, power structures.

Introduction

The aim of this paper is to emphasize the unifying role of Boolean algebras with operator. In the first section, we give a brief view of this concept and a list of theorems that play an important role in treating this concern. We start by recalling some notions, definitions and basic results about multialgebras, power structures, and Boolean algebras. In the second section, we introduce the concept of Boolean algebras with operator and we prove two important theorems regarding the relation between multialgebras and Boolean algebras with operator. Some theorems are taken without proof. There is a vast literature on this, but we cite [1,2,3,4].

The theory of Boolean algebras with operators has evolved from the work of Tarski on relational algebras.

Boolean algebra is the algebra of two-valued logic with only sentential connectives, or equivalently of algebras of sets under union and complementation. The rigorous concept is analogous to the mathematical notion of a group. Boolean algebra has applications in logic (Lindenbaum-Tarski algebras and model theory), set theory (fields of sets), topology (totally disconnected compact Hausdorff spaces), foundations of set theory (Boolean-valued models), measure theory (measure algebras), functional analysis (algebras of projections), and ring theory (Boolean rings) [3].

Multialgebras and Boolean Algebras with Operator

Definition 1. We say A is a multialgebra of type Φ with the set $A \neq \emptyset$ if A is a mapping from Φ to the family of multioperations of A . If $f \in \Phi_n$ then $A(f)$ is n -ary multioperation, which is interpretation of the functions' symbols in A and we denote it with f^A .

The set of all multialgebras of type Φ with the set A we will denote with $\Phi(A)$. Every multialgebra is considered as relational structure and vice versa, because we can set a bijection between the multioperations on A and the $(n+1)$ -ary relations on A .

Namely, to the relation $R \subseteq A^{n+1}$ responds the n -ary multioperation f^R given with:

$$y \in f^R(x_0, \dots, x_{n-1}) \Leftrightarrow (x_0, \dots, x_{n-1}, y) \in R$$

We note that the zero multioperation is a subset from A .

Since every multioperation f on A induces an operation f^+ on $\Pi(A)$, so, to every multialgebra $A \in \Phi(A)$ we associate an power structure $\Pi(A) = (\Pi(A), \{f^+ \mid f \in \Phi\})$.

Based on the concept of multialgebras and relying on the power structures of these algebras, which we could also consider as relational structures, and adding the set of operations $\cup, \cap, -$ we get a Boolean algebra with operator.

Definition 2. The power structure of relational structures $A = (A, P)$ is algebra with the set $\Pi(A)$ and the set of elementary operations $\{R^* \mid R \in R\}$ defined as follow:

for $R \in R_{n+1}$ we have $R^*: P(A)^n \rightarrow P(A)$ such that for $X_0, \dots, X_{n-1} \in P(A)$ the following properties applies $R^*(X_0, \dots, X_{n-1}) = \{y \in A \mid (\forall i \in \{0, \dots, n-1\})(\exists x_i \in X_i)(x_0, \dots, x_{n-1}, y) \in R\}$

It is obvious that for $R \in R_1 (n=0), R \subseteq A$, thus R^* is a zero operation which value is $R \in P(A)$.

We consider the universal algebras $A = (A, P)$ as relational structures, so that for each $f \in F_n$ we define the relation $R(f)$ of A , $(n+1)$ -ary such that $\forall a_0, \dots, a_{n-1}, a \in A$

$$(a_0, \dots, a_{n-1}, a) \in R(f) \Leftrightarrow f(a_0, \dots, a_{n-1}) = a.$$

Now, for the power structures holds the inclusion $X_0, \dots, X_{n-1} \in P(A)$,

$$R^*(f)(X_0, \dots, X_{n-1}) = \{x \in A \mid (\forall i \in \{0, \dots, n-1\})(\exists x_i \in X_i) f(x_0, \dots, x_{n-1}, y) = x\} = f^+(X_0, \dots, X_{n-1}).$$

We have the same situation for multialgebras. The difference stands that for the $(n+1)$ -ary relation of $R(f)$ in A , whom respond the multioperation $f \in F_n$, we define such that for each $\forall a_0, \dots, a_{n-1}, a \in A$,

$$(a_0, \dots, a_{n-1}, a) \in R(f) \Leftrightarrow a \in f(a_0, \dots, a_{n-1}),$$

Thus, the definition above we may consider as the definition of power structures of multialgebras.

Further, the power structure of relational structure $A = (A, P)$ we will denote with:

$$A^+ = (P(A), \cup, \cap, ^-, \{R^* \mid R \in R\})$$

As we previously mentioned that the structure $(B, +, \cdot, ', 0, 1)$ is Boolean algebra if "+" and "·" are binary operations, "' - unary operation where 0- zero and 1- one, hold the following properties:

(A1) $(B, +, \cdot)$ is distributive net.

(A2) $x \cdot 0 = 0, x + 1 = 1, \forall x \in B$.

(A3) $x \cdot x' = 0, x + x' = 1, \forall x \in B$.

If $\forall x, y \in B, x + y = y$, we denote $x \leq y$.

If for the set of elements $\{x_i \mid i \in I\}$ from B , exist the supremum and infimum, we denote them as $\Sigma\{x_i \mid i \in I\}$ and $\Pi\{x_i \mid i \in I\}$, respectively.

For any given set X , the algebra $B(X) = (P(A), \cup, \cap, ^-, \phi, X)$ is Boolean algebra.

If any nonempty closed subset related to all operations from B , is subalgebra of the Boolean Algebra B , then all subalgebras of the Boolean Algebra B are Boolean algebras as well. These sets of Boolean subalgebras we will call set algebras.

Theorem 1. [2] *Every Boolean algebra is isomorphic to any set algebra.*

Definition 3. *If B is a Boolean algebra, then the element $a \in B, a \neq 0$ is an atom if, for all $x \in B, x \cdot a = 0$ or $x \cdot a = a$. We denote with At_B the set of atoms of the Boolean algebra.*

We say a Boolean algebra is atomic if for all $x \in B, x \neq 0$ there exist the atom a such that $a \leq x$. The Boolean algebra is complete if every subset of B has infimum and minimum in B .

If the Boolean algebra is complete and atomic, then, $\forall x \in B$ we have $x = \Sigma\{a \in At_B \mid a \leq x\}$.

Theorem 2. *Every Boolean algebra can be divided into complete and atomic Boolean algebra.*

Proof. Let $(B, +, \cdot, ', 0, 1)$ be Boolean algebra, then the required complete and atomic Boolean algebra will be $B(B) = (P(B), \cup, \cap, ^-, \phi, B)$, while $id : B \rightarrow B(B), id(b) = \{b\}, \forall b \in B$.

On the other hand, every set of Boolean algebras is complete because the supremum, respectively, the infimum of any family of sets, indeed their union, respectively, their intersection is atomic, where the set of atoms consist the set of all subsets with a single element from $(At_B = \{\{b\} \mid b \in B\})$.

Definition 4. *Let B be Boolean algebra and $f : B^n \rightarrow B$. We say f is an operator if it is additive for every its argument, thus*

$$f(x_0, \dots, \Sigma\{y_i \mid i \in \{0, \dots, k-1\}\}, \dots, x_{n-1}) = \Sigma\{f(x_0, \dots, y_i, \dots, x_{n-1}) \mid i \in \{0, \dots, k-1\}\}$$

We say f is complete additive operator if f is an operator and for every set of indices I exist $\Sigma\{y_i \mid i \in I\}$ on B then $f(x_0, \dots, \Sigma\{y_i \mid i \in I\}, \dots, x_{n-1}) = \Sigma\{f(x_0, \dots, y_i, \dots, x_{n-1}) \mid i \in I\}$.

If f is an operator and $\forall a_0, \dots, a_{n-1} \in B$ applies $a_i = 0$ for some $i \in \{0, \dots, n-1\}$ then $f(a_0, \dots, a_{n-1}) = 0$, we say f is an normal operator.

Relating to the definitions given above we come to the concept of Boolean algebra with operator.

Definition 5. Boolean algebra with operator is the structure $B = (B, +, \cdot, ', 0, 1, \{f_i \mid i \in I\})$ where $(B, +, \cdot, ', 0, 1)$ is Boolean algebra or B -algebra, whereas the operations $f_i, i \in I$ are operators from B -algebra.

If the B -algebra is atomic, we say B is Boolean algebra with atomic operator and the set of atoms of B -algebra we shall denote with At_B . If B -algebra is complete and for every $i \in I, f_i$ is an complete additive operator we say B is Boolean algebra with complete operator. The Boolean algebra with operator which is complete, normal and atomic we will call fine Boolean algebra with operator.

Theorem 3.[1] Every normal Boolean algebra with operator can be divided into fine Boolean algebra with operator.

Theorem 4. The power structure of any multialgebra is fine Boolean algebra with operator.

Proof. Let the power structure of the relational structure (A, R) , $A^+ = (P(A), \cup, \cap, \bar{}, \phi, A, \{R^* \mid R \in R\})$ responds to the power structure of multialgebra $A \in F(A)$. Obviously, $(P(A), \cup, \cap, \bar{}, \phi, A)$ is atomic B -algebra while $R^*, \forall R \in R$ according to the definition is normal additive operator.

Definition 6. The atomic structure of a complete and atomic Boolean algebra with operator B , with the set of operators F , is the multialgebra $At_B = (At_B, \{\bar{f} \mid f \in F\})$, where for $f \in F_n$ and for all $b_0, \dots, b_{n-1} \in At_B$ the operation \bar{f} is defined as follows:

$$\bar{f}(b_0, \dots, b_{n-1}) = \{a \in At_B \mid a \leq f(b_0, \dots, b_{n-1})\}.$$

Theorem 5. Every fine Boolean algebra with operator is power structure of any multialgebra.

Proof. Let $B = (B, +, \cdot, ', 0, 1, F)$ be fine Boolean algebra with operator with atomic structure At_B and let $\varphi: B \rightarrow P(At_B)$ be a mapping defined as $\varphi(b) = \{a \in At_B \mid a \leq b\}$.

We will show that φ is an isomorphism from B to At_B^+ . Every element x from the complete and atomic Boolean algebra is of the form $x = \Sigma\{a \in At_B \mid a \leq x\}$, which holds even in this case. Hence, it comes that φ is a bijection.

We should also show that φ is homomorphism, thus, $\forall n \in N, f \in F_n, b_0, \dots, b_{n-1} \in B$ we have

$$\varphi(f(b_0, \dots, b_{n-1})) = \overline{f}^+(\varphi(b_0), \dots, \varphi(b_{n-1})).$$

The equation is satisfied trivially in the case when $b_i = 0$, for some $i \in \{0, \dots, n-1\}$, based on the definition of the normal operator and \overline{f}^+ we can now suppose that $b_i \neq 0$, $\forall i \in \{0, \dots, n-1\}$.

(\subseteq) Let $a \in \varphi(f(b_0, \dots, b_{n-1}))$. By the definition of mapping φ , it means that $a \in At_B$, $a \leq f(b_0, \dots, b_{n-1})$. We prove that $a \in \overline{f}^+(\varphi(b_0), \dots, \varphi(b_{n-1}))$, also for any $i \in \{0, \dots, n-1\}$, $\exists x_i \in \varphi(b_i)$ such that $a \in \overline{f}(x_0, \dots, x_{n-1})$ respectively, $a \leq f(x_0, \dots, x_{n-1})$.

Because $a \leq f(b_0, \dots, b_{n-1})$ and $b_i \in B$ for any $i \in \{0, \dots, n-1\}$, $b_i = \Sigma \varphi(b_i)$, $a \leq f(\varphi(b_0), \dots, \varphi(b_{n-1}))$ where $\varphi(b_i)$, $i \in \{0, \dots, n-1\}$ is a nonempty set because $b_i \neq 0$. Let be $\varphi(b_j) = \{x_{jk} \mid k \in I_j\}$. Since f is complete additive operator, we have:

$$f(\Sigma \varphi(b_0), \dots, \Sigma \varphi(b_{n-1})) = \Sigma \{f(x_{0k_0}, \dots, x_{(n-1)k_{n-1}}) \mid_{j \in \{0, \dots, n-1\}}^{k_j \in I_j}\}.$$

From the definition of the atom follows, $a \cdot f(x_{0k_0}, \dots, x_{(n-1)k_{n-1}}) = a$ or $a \cdot f(x_{0k_0}, \dots, x_{(n-1)k_{n-1}}) = 0$.

We show that $\exists x_i \in \varphi(b_i)$, $i \in \{0, \dots, n-1\}$ such that $a \leq f(x_0, \dots, x_{n-1})$, moreover $a \cdot f(x_0, \dots, x_{n-1}) = a$. By supposing the opposite that $\forall x_i \in \varphi(b_i)$, $i \in \{0, \dots, n-1\}$, $a \cdot f(x_0, \dots, x_{n-1}) = 0$, we get:

$$a \cdot f(b_0, \dots, b_{n-1}) = a \cdot \Sigma \{f(x_{0k_0}, \dots, x_{(n-1)k_{n-1}}) \mid_{j \in \{0, \dots, n-1\}}^{k_j \in I_j}\} = \Sigma a \cdot \{f(x_{0k_0}, \dots, x_{(n-1)k_{n-1}}) \mid_{j \in \{0, \dots, n-1\}}^{k_j \in I_j}\} = 0$$

Which leads to contradiction with the fact that $a \leq f(b_0, \dots, b_{n-1})$.

(\subseteq) Let $a \in \overline{f}^+(\varphi(b_0), \dots, \varphi(b_{n-1}))$. Hence $a \in At_B$ and $\exists x_i \in \varphi(b_i)$, $i \in \{0, \dots, n-1\}$ such that $a \leq f(x_0, \dots, x_{n-1})$.

From the additivity property of the operator f follows its monotony, while from $x_i \in \varphi(b_i)$ we get $x_i \leq b_i$, $\forall i \in \{0, \dots, n-1\}$, thus $a \leq f(x_0, \dots, x_{n-1}) \leq f(b_0, \dots, b_{n-1})$, also $a \in \varphi(f(b_0, \dots, b_{n-1}))$.

Conclusion

The study of Boolean algebras has several aspects: structure theory, model theory of Boolean algebras, decidability and undecidability questions for the class of Boolean algebras, and the indicated applications. Also, Boolean algebras with operator play main role in constructing power structures.

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