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THE UPPER NILRADICAL CONNECTED WITH A RIGHT IDEAL IN THE RING $\left(Z[\sqrt{p}], +, \cdot\right)$

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Abstract

Some properties of the rings vary from known properties, so we encounter difficulties in their classification. Herein lies the reason to seek the definition of a part of the ring about certain properties, the part that will be called a radical ring.

In addition to theoretical investigations of radicals connected with a right ideal in associative rings, we seek to build models implementing the achieved results. Based on the paper

The upper nilradical connected with a right ideal in the ring $\left(Z[\sqrt{3}], +, \cdot\right)$,in the process, we succeeded to construct an upper nilradical connected with a right ideal in the ring $\left(Z[\sqrt{p}], +, \cdot\right)$. This is presented in the paper, after some necessary theoretical preliminaries.

The study will be concentrated on the right ideal, since the theory of the left ideal of a ring is constructed analogously.

***Keywords**: The ring, right ideal, radical, nilradical, upper nilradical.*

1. The Meaning of Radical

Some properties of the rings vary from known properties, so we encounter difficulties in their classification. Herein lies the reason to seek the definition of a part of the ring about certain properties, the part that will be called a radical ring.

Let I_1, I_2 be sub rings of the ring *R*. With the sum, marked as $I_1 + I_2$, we understand the set that contains all sums $i_1 + i_2$, where $i_1 \in I_1, i_2 \in I_2$. So, $I_1 + I_2 = \{i_1 + i_2 / i_1 \in I_1, i_2 \in I_2\}$.

At first, let be examined these propositions:

Proposition 1.1. ([5], pg.397). If I_1, I_2 are ideals of the ring *R,* then

$$I_1 + I_2 = \{i_1 + i_2 / i_1 \in I_1, i_2 \in I_2\} \text{ is ideal of } R.$$

Let be now $I_1, I_2, ..., I_k, ...$ (not necessarily a finite number) a sub rings set of the ring *R*. With the sum, marked as $\sum_k I_k$, we understand the set that contains all sums

$i_1 + i_2 + ... + i_k + ...$, where $i_k \in I_k$, so, $\sum_k I_k = \{i_1 + i_2 + ... + i_k + ... \,/\, i_1 \in I_1, i_2 \in I_2, ...\}$.

Based on the above this result, we take this:

Proposition 1.2. ([2], pg.4). The sum of every ideal set of a ring is its ideal.

Let γ be a class of the rings such that:

a) γ is homomorphic closed: i.e. $A \in \gamma$ and $\varphi: A \to B$ is homomorphism, follows that $B \in \gamma$.

b) For every ring *A*, the sum $\gamma(A) = \sum (I \lhd A | I \in \gamma)$ is in γ.

c) $\gamma(A/\gamma(A)) = 0$, for every ring A.

Definition 1.1. A ring class γ that satisfies the conditions a), b), c), is called ***radical class***. $\gamma(A)$ is called γ -radical of *A*. The ring *A* is called γ **-radical ring** if $A \in \gamma \Rightarrow \gamma(A) = A$.

The conditions a), b), c) are essential conditions to define the radicals. To tell if these conditions are satisfied or not for some rings classes is not an easy task. For this reason we aim to find equivalent conditions with them.

Proposition 1.3. ([1], pg.22). If conditions a) and b) apply in a class rings γ, then the condition c) is equivalent with:

$\bar{c}$) If *I* is ideal of the ring *A* and if *I* and *A/I* are in γ, then the *A* is in γ.

When the class of the rings γ satisfies the condition $\bar{c}$) we say that γ is closed in connection to extensions.

Proposition 1.4. ([1], pg.23). If conditions a) and $\bar{c}$) apply in a rings class γ, then the condition b) is equivalent with:

$\bar{b}$) If $I_1 \subseteq I_2 \subseteq ... \subseteq I_\lambda \subseteq ...$ is an increasing sequence of ideals of a ring *A,* and if every of I_λ is in γ then $\sum_\lambda I_\lambda$ is in γ .

When the class of the rings γ satisfies the condition $\bar{b}$) we say that γ is with inductive property.

From the above propositions we get:

Theorem 1.1. ([1], pg.23). The class of rings γ is radical class then and only then when:

a) γ is homomorphic closed

$\bar{b}$) γ has the inductive property

$\bar{c}$) γ is closed in connection to extensions.

2. Nilrings and Nilradical

Definition 2.1. The element *x* of the ring *R* is called **nilpotent** if $\exists n \in N,\ x^n = 0$ (□ is the set of positive whole numbers).The ring *R* is called **nilring** if every element of it is nilpotent. The ideal *I* of the ring *R* is called **nilideal** of *R* if *I* is nilring.

We mark with *s* the property: $\forall x \in R,\ \exists n \in N,\ x^n = 0$ .The property *s* is called nil-property. In these conditions, we can say that **nilring** is the ring *R* that satisfies the nil-property *s*. Such a ring is called s-ring.

Lemma 2.1. ([2], pg.19).

a) If *R* is nilring then every subring and every homomorphic image of *R* is nilring.
b) If *I* is ideal of *R* and *I*, *R/I* are nilrings, then *R* is nilring.

Lemma 2.2. The sum of two nilideals is nilideal.

Proof. Let *I* and *J* be nilideals in *R*. Based on the second theorem on isomorphisms we have: $(I+J)/J \cong I/(I \cap J)$, where $(I+J)/J$ is nilideal from lemma 1.ii., whereas $I/(I \cap J)$ is nilideal as homomorphic image of *I*. Since *I, J,* $(I+J)/J$ are nilideals then it follows that *I+J* is nilideal.

Based on lemma 2.2 with mathematical induction is proved this:

Corollary 2.1. The finite sum of nilideals is nilideal.

From the above come true this:

Lemma 2.3. ([2], pg.19). The sum of a whatever set of nilideals of *R* is nilideal of *R*.

In the following we show that the class *S* of *s*-rings is radical class.

a) From lemma 2.1.i. follows that every homomorphic image of a s-ring is s-ring, i.e. the class *S* is homomorphic closed.

$\bar{c}$) From lemma 2.1.ii follows that, if *I* is ideal of *R* and *I*, *R/I* are in *S*, then *R* is in *S* as well. This shows that the class *S* is closed in connection to extensions.

b) From lemma 2.3 follows that, for every ring R, $S(R)=\sum(I \triangleleft R | I \in S)$ is in the class *S*.

Based on proposition 4 of the point 1 follows that the above condition b) is equivalent with:

$\bar{b}$) If $I_1 \subseteq I_2 \subseteq ... \subseteq I_\lambda \subseteq ...$ is an increasing sequence of nilideals of the ring R, then the sum is $\sum_\lambda I_\lambda$ in *S*.

As a result of the theorem of the point 1 we get this:

Theorem 2.1. The class *S* is radical class.

This theorem shows that *S*(*R*) is radical of the nilring *R*. *S*(*R*) is called nilradical of the ring *R*. From the above points *b*) and $\bar{b}$) follows that *S*(*R*) is larger nilideal of the ring *R*.

Definition 2.2. The largest nilideal of the nil ring *R* is called upper nilradical or nilradical of Köthe and is marked as *K*(*R*).

Conclusion 2.1 ([3], pg.47). In every ring *R*, the sum of a set of its nilideals is nilideal of *R*. Therefore, in every ring *R* exists the upper nilradical *K*(*R*) and complies with the sum of all nilideals of the ring *R*. If *I* is ideal in the ring *R,* then $K(I)=I \cap K(R)$. For every ring *R*, the factor-ring *R/K*(*R*)*,* there isn't nonzero nilideals, therefore $K(R/K(R))=0$.

Definition 2.3. The ring *R* is called *K*-radical, if *R=K (R)*. The ring *Q* is called *semi simple K*, if *K (Q)* = 0.

Theorem 2.2. ([3], pg.47). The ring R is K-radical then and only then, when it is nilring. The ring Q is *semi simple-K* then and only then, when it hasn't nonzero nilideals. Every ideal in the ring K-radical or *semi simple-K* is respectively ring *K-radical* or *semi simple-K*.

3. The Upper Nilradical Connected with a Right Ideal

Referring to the above treatment of nilradical of a ring, we presented a theoretical study on the meaning of the upper nilradical of the one-sided ideal of a ring. The study will be concentrated on the right ideal since the theory of the left ideal of a ring is constructed in an analogous way.

Let P be right ideal in the ring R.

Definition 3.1. ([6], pg.43) The ring R is called nilring of the right ideal P, if all of its elements are nilpotent of the right ideal P, i.e. $\forall a \in R, \exists n \in N, n \geq 2 \mid a^n \in P$.

The ideal I of ring R is called nilideal of the right ideal P, if I is nilring of the right ideal P. In analogous way is given the meaning of the nil subring of the right ideal P and the right (left) nilideal of the right ideal P.

Let R be a nilring of the right ideal P and ϕ a P-homomorphism in R. Since $\forall x \in R,\ \exists n \in \mathrm{N}, x^n \in P$ and $\phi\left(x^n\right) = \left[\phi(x)\right]^n \in P$, then P-homomorphic image $\phi(R)$ is nilring of the right ideal P.

We have proved this:

Theorem 3.1. Every P-homomorphic image of a nilring of the right ideal P is also nilring of the right ideal P.

Meanwhile are even worth these theorems:

Theorem 3.2. ([11], pg.78) If the ideal I is nilideal of the right ideal P in R and the factor-ring R/I is nilring, then R is nilring of the right ideal P.

Theorem 3.3. ([4], pg.114). The sum of two nilideals of the right nilideal P is nilideal of the right ideal P. [9]

Based on proposition 4 of point 1 derives that the above condition c) is equivalent with: $\bar{c}$) If $I_1 \subseteq I_2 \subseteq ... \subseteq I_\lambda \subseteq ...$ is an increasing sequence of nilideals of a right ideal P of A then the sum $\sum_\lambda I_\lambda$, is nilideal of the right ideal P.

Having in mind the meaning of the Köthe nilradical we give this:

Definition 3.2. Upper nilradical of the right ideal P in the ring R is called the largest nilideal of the right ideal P in it. It is marked as $K(R, P)$.

Conclusion 3.1. In a ring R, the sum of every finite set of nilideals of the right ideal P is nilideal of the right ideal P. After, in every ring R, exists the upper nilradical $K(R, P)$ of the right ideal P and it complies with the sum of all nilideals of the right ideal P. If I is ideal of the right ideal P of ring R, then $K(I,P) = I \cap K(R,P)$.

Definition 3.3. The ring R is called K-radical of the right ideal P, if $R = K(R,P)$.

4. The Upper Nilradical connected with a right ideal in the Ring $(\mathrm{Z}[\sqrt{\mathrm{p}}], +, \cdot)$

Let $\mathrm{Z}[\sqrt{\mathrm{p}}]$ be a set of numbers $x = a + b\sqrt{p}$ such that a,b are whole numbers and p is a prime number different from 2, so

$$\mathrm{Z}[\sqrt{\mathrm{p}}] = \{a + b\sqrt{p} \mid a,b \in \mathrm{Z}\}$$

Since,

- $x + y = (a + b\sqrt{p}) + (c + d\sqrt{p}) = (a + c) + (b + d)\sqrt{p} \in Z[\sqrt{p}]$ and
- $xy = (a + b\sqrt{p}) \cdot (c + d\sqrt{p}) = (ac + pbd) + (ad + bc)\sqrt{p} \in Z[\sqrt{p}]$.

  Imply $(\mathrm{Z}[\sqrt{\mathrm{p}}], +, \cdot)$ is algebraic structure, and since
- $x - y = (a + b\sqrt{p}) - (c + d\sqrt{p}) = (a - c) + (b - d)\sqrt{p} \in Z[\sqrt{p}]$.

  Follows that $(\mathrm{Z}[\sqrt{\mathrm{p}}], +, \cdot)$ is sub ring of the ring R of the real numbers.

We note $P[\sqrt{p}]$ the set $\{a + b\sqrt{p} \mid a = 2r, b = 2s \wedge r,s \in \mathrm{Z}\}$.

It is clear that $p \in P[\sqrt{p}] \Leftrightarrow p = 2(r + s\sqrt{p}),\ where\ r,s \in Z \Leftrightarrow p = 2x,\ where\ x \in Z[\sqrt{p}]$.
Therefore, $P[\sqrt{p}\,] = \{2x \mid x \in Z[\sqrt{p}\,]\}$.

Proposition 4.1. The set $P[\sqrt{p}]$ is right ideal of the ring $(\mathrm{Z}[\sqrt{\mathrm{p}}], +, \cdot)$.

Indeed, since $Z[\sqrt{p}]$ is ring we have:

1. $\forall p_1, p_2 \in P[\sqrt{p}]$, i.e. $p_1 = 2x_1$, $p_2 = 2x_2$, where $x_1, x_2 \in Z[\sqrt{p}]$, we get

   $p_1 - p_2 = 2(x_1 - x_2) \in P[\sqrt{p}]$, because $x_1 - x_2 \in Z[\sqrt{p}]$

2. $\forall p \in P[\sqrt{p}], \forall y \in Z[\sqrt{p}],\ p \cdot y = 2xy \in P[\sqrt{p}]$, because $xy \in Z[\sqrt{p}]$.

We note N $\left(Z[\sqrt{p}], P[\sqrt{p}]\right)$ the set of numbers $x \in Z[\sqrt{p}]$ such that $x^n \in P[\sqrt{p}]$ for every natural number $n \geq 2$, i.e.

$$\mathrm{N}\ \left(Z[\sqrt{p}], P[\sqrt{p}]\right) = \{\, x \in Z[\sqrt{p}] \mid \forall n \in N,\ n \geq 2,\ x^n \in P[\sqrt{p}]\, \} \qquad (1)$$

Proposition 4.2. The set $P[\sqrt{p}]$ is included strictly to the set

$$\mathrm{N}\ \left(Z[\sqrt{p}], P[\sqrt{p}]\right), \text{ so } P[\sqrt{p}] \subset \mathrm{N}\ \left(Z[\sqrt{p}], P[\sqrt{p}]\right).$$

This since:

1. $\forall p \in P[\sqrt{p}],\ \forall n \in N,\ n \geq 2,$ we have $p^n = (2x)^n = 2(2^{n-1}x^n) = 2y \in P[\sqrt{p}]$,
   because being $Z[\sqrt{p}]$ the ring, $y = 2^{n-1}x^n \in Z[\sqrt{p}]$. So,

$$\forall\, p \in P[\sqrt{p}],\ p^n \in P[\sqrt{p}] \ \text{ for } \ \forall n \in \mathrm{N},\ n \geq 2. \qquad (2)$$

According to (1) this implies $p \in \mathrm{N}\ \left(Z[\sqrt{p}], P[\sqrt{p}]\right)$, i.e.

$$P[\sqrt{p}] \subseteq \mathrm{N}\ \left(Z[\sqrt{p}], P[\sqrt{p}]\right).$$

2. We take $x = a + b\sqrt{p} \in Z[\sqrt{p}]$, where $a = 2r+1, b = 2s+1$ *and* $r, s \in Z$.
   Since *a, b* aren't even numbers, so $x \notin P[\sqrt{p}]$. However $x^2 \in P[\sqrt{p}]$, because

$$x^2 = \left(a + b\sqrt{p}\right)^2 = a^2 + 2ab\sqrt{p} + pb^2 =$$

$$= (2r+1)^2 + 2(2r+1)(2s+1)\sqrt{p} + p(2s+1)^2 = = 4r^2 + 4r + 1 + 2(2r+1)(2s+1)\sqrt{p} + p(4s^2 + 4s + 1) =$$

$$= 4r^2 + 4ps^2 + 4r + 4s + p + 1 + 2(2r+1)(2s+1)\sqrt{p} =$$

Since p is odd, then $p + 1 = 2l$ is an even and therefore we have

$$= 2[(2r^2 + 6s^2 + 2r + 2s + l) + (2r+1)(2s+1)\sqrt{p}] \in P[\sqrt{p}].$$

With mathematical induction is proved that: $x^n \in P[\sqrt{p}]$ për $\forall n \in \square,\ n \geq 2.$ So,

$$\forall\ x = (2r+1) + (2s+1)\sqrt{p} \in Z[\sqrt{p}],\ x^n \in P[\sqrt{p}] \text{ për } \forall n \in \square,\ n \geq 2. \qquad (3)$$

From (2) and (3), derives that:

$$P[\sqrt{p}] \subset \mathrm{N}\ (Z[\sqrt{p}], P[\sqrt{p}]). \qquad (4)$$

Proposition 4.3. In N $(Z[\sqrt{p}],P[\sqrt{p}])$ take part only two type of numbers from $Z[\sqrt{p}]$, and exactly 1) numbers of the form $a+b\sqrt{p}$, where $a=2r, b=2s$ and $r,s\in Z$

2) numbers of the form $a+b\sqrt{p}$, where $a=2r+1, b=2s+1$ and $r,s\in Z$.

It is clear that in the ring $Z[\sqrt{p}]$ take part these type numbers:

1. $a+b\sqrt{p}$, where $a=2r, b=2s$ and $r,s\in Z$;
2. $a+b\sqrt{p}$, where $a=2r+1, b=2s+1$ and $r,s\in Z$;
3. $a+b\sqrt{p}$, where $a=2r+1, b=2s$ and $r,s\in Z$;
4. $a+b\sqrt{p}$, where $a=2r,\ b=2s+1$ and $r,s\in Z$.

From (3) derives that numbers of type 1) and 2) are numbers in N $(Z[\sqrt{p}],P[\sqrt{p}])$, whereas numbers of type 3) and 4), don't take part in N $(Z[\sqrt{p}],P[\sqrt{p}])$, this could be proved in direct way.

Proposition 4.4. The set N $(Z[\sqrt{3}],P[\sqrt{3}])$ is the right ideal of ring $Z[\sqrt{3}]$ of its right ideal

$P[\sqrt{3}]$.

Proof .1 .For every $z_1, z_2\in$ N $(Z[\sqrt{p}],P[\sqrt{p}])$ that are of the first type, we have:

$z_1 - z_2 = (2r_1+2s_1\sqrt{p})-(2r_2+2s_2\sqrt{p}) = 2(r_1-r_2)+2(s_1-s_2)\sqrt{p} =$

$=2[(r_1-r_2)+(s_1-s_2)\sqrt{p}]=2x\in P[\sqrt{p}]$.

For every $z_1, z_2\in$ N $(Z[\sqrt{p}],P[\sqrt{p}])$ that are of the second type, we have

$z_1 - z_2 = (2r_1+1)+(2s_1+1)\sqrt{p} -[(2r_2+1)+(2s_2+1)\sqrt{p}]=$

$=2[(r_1-r_2)+(s_1-s_2)\sqrt{p}]=2x\in P[\sqrt{p}]. = 2(r_1-r_2)+2(s_1-s_2)\sqrt{p} \quad =2x\in P[\sqrt{p}]$

If z_1 is of the first type and z_2 is of the second type i.e.

$z_1 = 2r_1+2s_1\sqrt{p}$ and $z_2=(2r_2+1)+(2s_2+1)\sqrt{p}$, then

$z_1-z_2=[2(r_1-r_2-1)+1]+[2(s_1-s_2-1)+1]\sqrt{p} = (2r_3+1)+(2s_3+1)\sqrt{p}$, being number of the second type takes part in N $(Z[\sqrt{p}],P[\sqrt{p}])$. Finally, if z_1 is of the second type and z_2 is of the first type, in an analogous way could be said again $z_1-z_2\in$N $(Z[\sqrt{p}],P[\sqrt{p}])$.

As conclusion, $\forall z_1,z_2\in$ N $(Z[\sqrt{p}],P[\sqrt{p}])$, $z_1-z_2\in$ N $(Z[\sqrt{p}],P[\sqrt{p}])$.

1. Also, $\forall z \in \mathrm{N}\left(Z[\sqrt{p}], P[\sqrt{p}]\right)$, $\forall x \in Z[\sqrt{p}]$ follows that $zx \in \mathrm{N}\left(Z[\sqrt{p}], P[\sqrt{p}]\right)$. This since $z \in \mathrm{N}\left(Z[\sqrt{p}], P[\sqrt{p}]\right)$ and z is of first type, then we have $z = 2r + 2s\sqrt{p}$,where $r, s \in Z$ and since $x \in Z[\sqrt{p}]$,we have $x = a + b\sqrt{p}$, where $a, b \in Z$.

   Then, $zx = 2(ar + psb) + 2\left(br + as\sqrt{p}\right) = 2m + 2n\sqrt{p} \in \mathrm{N}\left(Z[\sqrt{p}], P[\sqrt{p}]\right)$.

   Now, we get the case when z is of second, so, $z = (2r+1) + (2s+1)\sqrt{p}$, where $r, s \in Z$ and since $x \in Z[\sqrt{p}]$, we have $x = a + b\sqrt{p}$ where $a, b \in Z$.

   $$zx=[(2r+1) + (2s+1)\sqrt{p}](a+b\sqrt{p}) = =(2ra+a+2sbp+bp)+(2rb+b+2sa+a)\sqrt{p}.$$

   Being even or odd number of expressions $2ra+a+2sbp+bp$ and $2rb+b+2sa+a$ depends on $a + pb$, respectively $a + b$, then we get cases:

i) When a, b are even numbers derives that $a + pb$ respectively $a + b$ is even, hence zx is of first type, so $zx \in \mathrm{N}\left(Z[\sqrt{p}], P[\sqrt{p}]\right)$.

ii) When a, b are odd numbers derives that $a + pb$ respectively $a + b$ is even, hence zx is of second type, so $zx \in \mathrm{N}\left(Z[\sqrt{p}], P[\sqrt{p}]\right)$.

iii) When a is odd number and b is even number derives that $a + pb$ respectively $a + b$ is odd, hence zx is of second type, so $zx \in \mathrm{N}\left(Z[\sqrt{p}], P[\sqrt{p}]\right)$.

iv) When a is even number and b is odd number derives that $a + pb$ respectively $a + b$ is odd, hence zx is of second type, so $zx \in \mathrm{N}\left(Z[\sqrt{p}], P[\sqrt{p}]\right)$.

We get $\forall z \in \mathrm{N}\left(Z[\sqrt{p}], P[\sqrt{p}]\right)$ and $\forall x \in Z[\sqrt{p}]$ we have: $zx \in \mathrm{N}\left(Z[\sqrt{p}], P[\sqrt{p}]\right)$.

From formula (1) and the proposition 4 derives that $\mathrm{N}\left(Z[\sqrt{p}], P[\sqrt{p}]\right)$ is right nil ideal of ring $Z[\sqrt{p}]$ of the right ideal $P[\sqrt{p}]$.

Beside subset $\mathrm{N}\left(Z[\sqrt{p}], P[\sqrt{p}]\right)$ of ring $Z[\sqrt{p}]$ we examine the subsets:

$$I_1 = \{a + b\sqrt{p} \mid a = 2r,\ b = 2s \ and \ r, s \in Z\}$$

$$I_2 = \{a + b\sqrt{p} \mid a = 2r+1,\ b = 2s+1 \ and \ r, s \in Z\}.$$

Since $I_1 = P[\sqrt{p}]$, from proposition 4.2 derives that I_1 is right nilideal of ring $Z[\sqrt{p}]$ of the $P[\sqrt{p}]$. Since $I_1 + I_2 = \mathrm{N}\left(Z[\sqrt{p}], P[\sqrt{p}]\right)$, from proposition 4.4 and $I_1 + I_2$ is right nil ideal of ring $Z[\sqrt{p}]$ of the $P[\sqrt{p}]$. Having in mind the proposition 4.3, they are only ones such nilideals.

Conclusion

As conclusion, the nilideal N $\left(Z[\sqrt{p}], P[\sqrt{p}]\right)$ *is the sum of all right nilideals* of ring $Z[\sqrt{p}]$ of the right ideal $P[\sqrt{p}]$. Thus, the largest nilideal of ring $Z[\sqrt{p}]$ of a right ideal $P[\sqrt{p}]$, according to the definition of the upper nilradical of a right ideal, N $\left(Z[\sqrt{p}], P[\sqrt{p}]\right)$ **is upper nilradical of ring** $Z[\sqrt{p}]$ **of a right ideal** $P[\sqrt{p}]$.

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