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FIXED POINT THEOREMS IN FUZZY METRIC SPACES

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Abstract

In this paper are given some features and important results in the fuzzy metric space. The results that are presented are related to the fixed point in the fuzzy metric space. The theorems which are elaborated provide the existence and uniqueness of the fixed point in the complete fuzzy metric space. The proof of which is done by using Cauchy sequence. Also are given results for the fixed point in the compact fuzzy metric spaces. All these results are explained by concrete examples that once also present their application.

Keywords: *fuzzy set, fuzzy metric space, complet fuzzy metric speces, compact fuzzy metric spaces, continuous t-norm, fixed point*

1. Introduction

The concept of fuzzy sets was initiated by L. Zadeh 1965 [5]. This concept was used in topology and analysis by many authors. The concept of fuzzy metric space was introduced by Kramosil and Michalek [3]. Later on, George and Veeramani gives the modified notion of fuzzy metric spaces due to Kramosil and Michalek [3] and analyzed a Hausdorff topology of fuzzy metric spaces. In 1988, Grabiec [4] proved an analog of the Banach contraction theorem in fuzzy metric spaces. In his proof, he used a fuzzy version of the Cauchy sequence.

2. Preliminaries

To initiate the concept of a fuzzy metric space, which was introduced by Kramosil and Michalek [3] in 1975 is recalled here.

Definition 2.1: A fuzzy set $\tilde{A}$ is defined by $\tilde{A} = \{(x, \mu A(x)) : x \in A, \mu A(x) \in [0,1]\}$.

In the pair $(x, \mu A(x))$, the first element x belongs to the classical set A, the second element $\mu A(x)$ belongs to the interval [0,1], is called the membership function.

Definition 2.2: A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if it satisfies the following conditions:

(1) $*$ is associative and commutative,

(2) $*$ is continuous,

(3) $a * 1 = a$ for all $a \in [0,1]$,

(4) $a * b \le c * d$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0,1]$

Examples 2.1:

i. Lukasievicz t-norm: $a * b = max\{a + b - 1,0\}$
ii. Product t-norm: $a * b = a.b$
iii. Minimum t-norm: $a * b = min(a, b)$

Definition 2.3: A fuzzy metric space is an ordered triple $(X, M,*)$ such that X is a nonempty set, $*$is a continuous t-norm and M is a fuzzy set on $X \times X \times (0, \infty) \to [0,1]$ satisfies the following conditions: $\forall x, y, z \in X$ and $s, t > 0$
(KM1) $M(x, y, 0) = 0$,
(KM2) $M(x, y, t) = 1$, iff $x = y$, $t > 0$
(KM3) $M(x, y, t) = M(y, x, t)$
(KM4) $M(x, z, s + t) \geq M(x, y, s) * M(y, z, t)$
(KM5) $M(x, y, .): [0,\infty) \to [0,1]$ is left-continuous.
Then M is called a fuzzy metric on X.

Example 2.2: Let (X, d) be a metric space. Define $a * b = ab(a * b = min\{a, b\})$ for all $a, b \in [0,1]$ and define $M: X \times X \times (0, \infty) \to [0,1]$ as

$$M(x, y, t) = \frac{t}{t + d(x, y)}$$

$\forall x, y \in X$ dhe $t > 0$. Then $(X, M,*)$ is a fuzzy metric space.
We call this fuzzy metric induced by the metric d the standard fuzzy metric.

Definition 2.4: Let $(X, M,*)$ be a fuzzy metric space, for $t > 0$ the open ball $B(x, r, t) \subset A$ with a centre $x \in X$ and a radius $0 < r < 1$ is defined by

$$\mathrm{B}(x, r, t) = \{y \in X: M(x, y, t) > 1 - r\}$$

A subset $A \subset X$ is called open if for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subset A$. Let τ denote the family of all open subsets of X. Then τ is topology on X, called the topology induced by the fuzzy metric M.

Definition 2.5: Let $(X, M,*)$ be a fuzzy metric space
(i) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if $\lim_{n\to\infty} M(x, y, t) = 1$ for all $t > 0$.
(ii) A sequence $\{x_n\}$ in X is called a Cauchy sequence, if for each $0 < \epsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \epsilon$ for each $n, m \geq n_0$.
(iii) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.
(iv) A fuzzy metric space in which every sequence has a convergent subsequence is said to be compact.

Lemma 2.1: Let $(X, M,*)$ be a fuzzy metric space. For all $x, y \in X, M(x, y, .)$ is non-decreasing function.
Proof: If $M(x, y, t) > M(x, y, s)$ for some $0 < t < s$.
Then $(x, y, t) * M(y, y, s - t) \leq M(x, y, s) < M(x, y, t)$,
Thus, $(x, y, t) < M(x, y, t)$, (since $M(y, y, s - t) = 1$)

which is a contradiction.

Theorem 2.1: (fuzzy Edelstein contraction theorem) Let $(X, M,*)$ be a compact fuzzy metric space with $M(x, y,.)$ continuous for all $x, y \in X$. Let $T: X \to X$ be a mapping satisfying

$$M(Tx, Ty, t) > M(x, y, t) \quad \ldots(2.1)$$

for $x \neq y$ and $t > 0$. Then T has a unique fixed point p in X.

Theorem 2.2: Let $(X, M,*)$ be a complete fuzzy metric space and $T: X \to X$ is continuous function and satisfied the condition:

$$M(Tx, Ty, t) \geq min\{M(Tx, y, t), M(x, Ty, t), M(x, y, t)\} \quad \ldots(2.2)$$

Moreover, the fuzzy metric$M(x, y, t)$ satisfies the condition

$$\lim_{n\to\infty} M(x, y, t) = 1$$

Where $x, y \in X$ and $x \neq y$, then T has a fixed point in X.

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Moreover, the fuzzy metric $M(x, y, t)$ satisfies the condition

$$\lim_{n\to\infty} M(x, y, t) = 1$$

Where $x, y \in X$ and $x \neq y$. Then T has a fixed point in X.

3. Main Results

Corollary 3.1: Let $(X, M,*)$ be a complete fuzzy metric space and $T^n: X \to X, n \in \mathbb{N}$ is continuous function and satisfied the condition:

$$M(T^n x, T^n y, t) \geq min\{M(T^n x, y, t), M(x, T^n y, t), M(x, y, t)\} \quad \ldots(3.1)$$

Moreover, the fuzzy metric$M(x, y, t)$ satisfies the condition

$$\lim_{n\to\infty} M(x, y, t) = 1$$

Where $x, y \in X$ and $x \neq y$. Then T has a fixed point in X.

Proof. From theorem 2.1, T^n has a unique fixed point p in , so $T^n p = p$.
Since $Tp = TT^n p = T^n Tp$, Tp is also a fixed point of T^n. By the uniqueness it follows $Tp = p$.

Corollary 3.2 : Let $(X, M,*)$ be a complete fuzzy metric space and $T^n: X \to X, n \in \mathbb{N}$, is continuous function and satisfied the condition:

$$M(T^n x, T^n y, t) \geq min\{M(x, T^n x, t), M(y, T^n y, t), M(x, y, t)\} \quad \ldots(3.2)$$

Moreover, the fuzzy metric$M(x, y, t)$ satisfies the condition

$$\lim_{n\to\infty} M(x, y, t) = 1$$

Where $x, y \in X$ and $x \neq y$. Then T has a fixed point in X.

Proof. The proof is similar to that of corollary 3.1.

Example 3.1: Let $X = [0,1]$ and fuzzy metric is defined by $M(x, y, t) = \frac{t}{t+|x-y|}$, $t > 0$ and* be the continuous $t-$norm defined by $a * b = min(a, b)$.

Conclusion

Clearly, $(X, M,*)$ is a complete fuzzy metric space and T be a self map on X is given by $Tx = 1 - x$.

Now, (3.1) and (3.2) of Corollary 3.1 and Corollary 3.2 are satisfied by above defined fuzzy metric and T.

Thus T has a unique fixed point in X. That is at $= \frac{1}{2}$.

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