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INVERSE SEMIGROUP AS A SPECIAL CLASS OF REGULAR SEMIGROUPS

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Abstract

In this paper, we have studied some properties of inverse semigroups, which play an important role in the semigroup theory. Firstly, we give some general notations, definitions and auxiliary facts related to inverse semigroups. Then we study the close relationship of inverse and regular semigroup and the double inverse semigroup as a special class of inverse semigroups.

Keywords: *Semigroup, inverse, regular, double integral.*

1. Introduction

The idea of inverse semigroups or generalized groups was introduced, independently by Preston and Vagner and has been studied by other mathematicians, such as Clifford, Ljapin, Munn, and Penrose. The following theorem demonstrates a connection between regular and inverse semigroups. By a further investigation of double inverse semigroups, we can show that the two operations of any double inverse semigroups must coincide and thus double inverse semigroups are commutative inverse semigroups.

We recall the definition, properties of regular and inverse semigroups and referring to [1], [4] and [5] for the proofs of theorems. Also, we refer to [2] and [6] for definitions and properties about double inverse semigroup.

Firstly, we shall define some classical definitions and lemmas which will be used throughout the paper.

2. Relation of inverse semigroup with regular semigroup

Definition 2.1 A semigroup is a set S together with a binary operation " • " that satisfies the associative property: For all $a,b,c \in S$ the equation $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ holds.

Definition 2.2 An **inverse semigroup** (an inversion semigroup) S is a semigroup in which every element x in S has a unique inverse y in S in the sense that $x = xyx$ and $y = yxy$ Clearly, all groups are inverse semigroups.

Definition 2.3 A semigroup S is said to be regular if for each $x \in S$ there exists an element y such that $x = xyx$ and $y = yxy$. The element y is said to be an inverse of x. Thus inverse semigroups are the regular semigroups in which each element has a unique inverse.

Definition 2.4 A principal right [left] ideal of a semigroup S is an ideal of the form aS^1, where $aS^1 = aS \cup a$ and $a \in S\left[S^1 a,\ a \in S\right]$.

Definition 2.5 A monoid is a semigroup with an identity element. The notation S^1 denotes a monoid obtained from S by adjoining an identity *if necessary*.

Definition 2.6 An idempotent in a semigroup S is an element e such that $e^2 = e$. Idempotents play an important role in inverse semigroup. The set of idempotents of S is denoted by $E(S)$. Two special idempotents are the identity element if it exists, and the zero elements, if it exists.

Definition 2.7 An inverse semigroup with an identity element is called an inverse monoid and an inverse semigroup with zero is called an inverse semigroup with zero. An inverse subsemigroup of an inverse semigroup is a subsemigroup that is also closed under inverses.

Proposition 2.1 A *regular semigroup is inverse if and only if its idempotents commute.*

Proof: Let S be a regular semigroup in which the idempotents commute and let u and v be inverses of x. Then,

$$u = uxu = u(xvx)u = (ux)(vx)u,$$

where both ux and vx are idempotents. Thus, since idempotents commute, we have that

$$u = (vx)(ux)u = vxu = (vxv)xu = v(xv)(xu)$$

Again, xv and xu are idempotents and so

$$u = v(xu)(xv) = v(xux)v = vxv = v. \text{ Hence } u = v.$$

The converse is a little trickier. Observe first that in a regular semigroup the product of two idempotents e e and f has an idempotent inverse. To see why, let $x = (ef)'$ be any inverse of ef. Then the element fxe is an idempotent inverse of ef. Now let S be a semigroup in which every element has a unique inverse. We shall show that $ef = fe$ for any idempotents e and f. By the result above, $f(ef)'e$ is an idempotent inverse of ef. Thus $(ef)' = f(ef)'e$ by uniqueness of inverses, and so $(ef)'$ is an idempotent. Every idempotent is self-inverse, but on the other hand, the inverse of $(ef)'$ is ef. Thus $ef = (ef)'$ by uniqueness of inverses. Hence ef is an idempotent. We have shown that the set of idempotents is closed under multiplication. It follows that fe is also an idempotent.

But $ef(fe)ef = (ef)(ef) = ef$ and $fe(ef)fe = fe$ since ef and fe are idempotents. Thus ef and fe are inverses of ef. Hence $ef = fe$.

Inverses in inverse semigroups behave much like inverses in groups.

Proposition 2.2 All groups are inverse semigroups and an inverse semigroup is a group if and only if it has a unique idempotent.

Proof: Clearly, groups are inverse semigroups. Conversely, let S be an inverse semigroup with exactly one idempotent, e says. Then $s^{-1}s = e = ss^{-1}$ for each $s \in S$. But $es = (ss^{-1})s = s = s(s^{-1}s) = se$ and so e is the identity of S. Hence S is a group.

Lemma 2.1 An element a of a semigroup is regular if and only if the principal right [left] ideal of S generated by a has an idempotent generator, that is $aS^1 = eS^1\left[S^1a = eS^1\right]$.

Proof: Suppose a is regular too show $aS^1 = eS^1$. For a there exists $x \in S$ such that $axa = a$ and $axax = ax = e$. For $y \in aS^1$ if $y = a$ then $a = axa$ which implies that $a \in axS^1$. If $y \neq a$ there exists $p \in S$ such that $y = ap = axap$. Now $ap \in S$, so $y \in axS^1$. Similarly for $S^1a = S^1xa = S^1e$ where $xa = e$.

Supose that $aS^1 = eS^1$ and $S^1a = S^1e$ for some idempotent e ∈ S. Therefore $a = et$ or $a = e$ and $e = aq$, so $a = et = eet = (aq)et = (aq)a$ or $a = e = eee = aaa$. Therefor $a = aqa$ or $a = aaa$ and a a is a regular element.

Lemma 2.2 If a a is a regular element of a semigroup S, with x ∈ S then a has at least one inverse in S, in particular xax.

Proof: $axa = a$ which implies $axaxa = axa = a$ or $a(xax)a = a$. Now
$xax = xaxax = xaxaxax = (xax)a(xax)$

So a and xax are inverse elements of each other.

Lemma 2.3 e, f, ef, fe are idempotents of a semigroup then ef and fe are inverses of each other.

Proof: $(ef)(ef)(ef) = (eff)(eef) = ef * ef = ef$

$$(fe)(ef)(fe) = (fee)(ffe) = fe * fe = fe$$

Theorem 2.1 S is an inverse semigroup if and only if S is regular and for any idempotent $e \in S$, the equations $exe = e$ and $xex = x$ have no common idempotent solution other than e.

Proof: If S is an inverse semigroup then S is regular and since e is its own unique inverse then $exe = e$ and $xex = x$ have a unique solution.

Suppose S is regular and $exe = e$ and $xex = x$ have a unique common solution. We have shown that every element a in a regular semigroup has an inverse x and so $axa = a$ which imply that an inverse of a is xax Therefore there exists $b \in S$ such that $aba = a$ and $bab = b$. Suppose there exists b' ∈ S $b' \in S$ such that $b \neq b'$ b ≠ b', $ab'a = a$ and b'ab' = b' $b'ab' = b'$. Now $(ba)(ba) = ba$ and $(b'a)(b'a) = b'a$ so $(b'a)(b'a) = b'a = (b'aba)b'a = (b'a)(ba)(b'a)$ and

$ba = (ba)(ba\_ = (bab^{'}a)ba = (ba)(b^{'}a)(ba)$ which implies $b^{'}a = ba$. Likewise for (ab)2 = ab and (ab')2 = ab', ab = abab = ab(ab'ab) = (ab)(ab')(ab) and ab' = (ab')(ab') = (ab')(abab') = (ab')(ab)(ab'), which implies ab = ab'. Now b = bab = bab' and b' = b'ab' = bab', which implies b = b'. Therefore, S is an inverse semigroup by definition.

3. Inverse double semigroup and its commutativity

Definition 3.1 A double semigroup (S , ⊚, ⊙) is a set equipped with two associative binary operations ⊚ *and* ⊙ such that, for all a,b,c,d ∈ S, the following law, called the middle-four interchange, holds:

$(a ⊚ b) ⊙ (c ⊚ d) = (a ⊙ c) ⊚ (b ⊙ d)$.

Definition 3.2 A double inverse semigroup (D, ⊚,⊙) is a double semigroup in which both operations are inverse semigroup operations. It is shown by Kock that all double inverse semigroups must be commutative.

Definition 3.3 A semigroup S is said to be right cancellative if for any $a,b,c \in S$, $ac = bc$ implies $a = b$. S is said to be left cancellative if for any $a,b,c \in S$, $ca = cb$ implies $a = b$. We say that S is cancellative if S is both left cancellative and right cancellative. A double semigroup is said to be cancellative if both of its operations are.

Notation: If (S , ⊙) is an inverse semigroup and $a \in S$, we denote the semigroup inverse of a by $a^{⊙}$.

Proposition 3.1 Let (S, ⊙) be an inverse semigroup. Then $s^{⊙}$s, s $s^{⊙}$ ∈ E(S) for all $s \in S$.

Proof: We see that ($s^{⊙}$s)($s^{⊙}$s) = ($s^{⊙}$s$s^{⊙}$)s = $s^{⊙}$s and thus $s^{⊙}$s is an idempotent.

Similarly, (s $s^{⊙}$)(s $s^{⊙}$) = s $s^{⊙}$.

Proposition 3.2

Notation: If (D ,⊚,⊙) is a double semigroup, we can assign the operations each their own respective direction. That is, we can consider ⊚ and ⊙ as a vertical operation. This is useful because it provides an easily interpreted visualisation of products in D:

- For any $a,b \in D$, we represent the product a ⊚ b horizontally as :

A	B

- For any $a,b \in D$, we represent the product a ⊙ b vertically as :

A
B

It is now noted that we can interpret products such as:

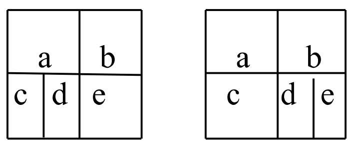
A	B
C	D

without any ambiguity because the middle four interchange law implies equality in choice of operational order (it does not matter whether we evaluate the horizontal product of the two vertical products or vice versa).

Because both ⊚ and ⊙ are associative operations, we can rewrite large products of elements in D in several ways. If, for example, we consider the product (a ⊚ b) ⊙ (c ⊚ d ⊚ e) we can rewrite this as either

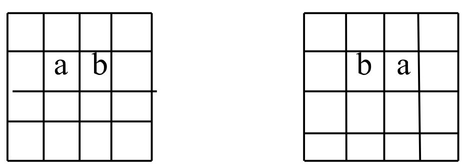
(a ⊚ b) ⊙ ((c ⊚ d) ⊚ e) or (a ⊚ b) ⊙ (c ⊚ (d ⊚ e)).

Visually, one has that



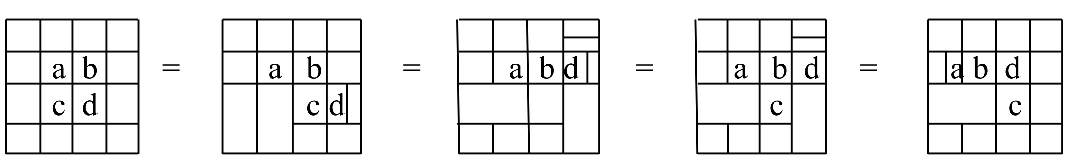
As an immediate consequence of the definition of a double semigroup, Kock establishes the following commutativity result:

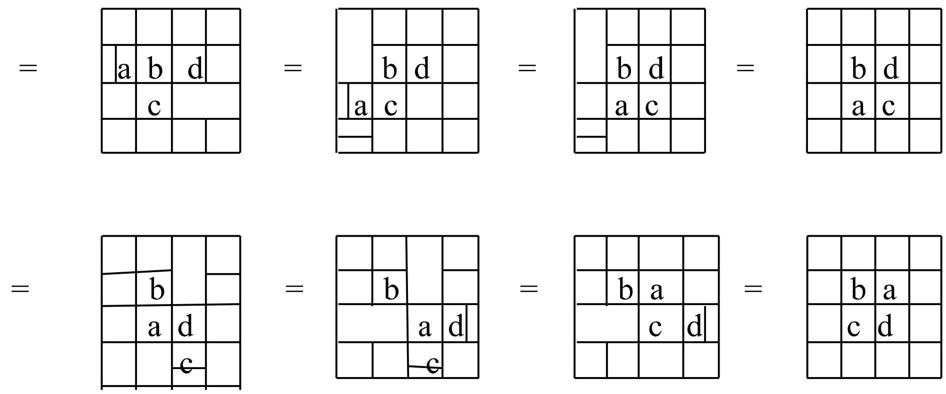
Theorem 3.1 For any sixteen elements a, b, ... in any double semigroup, this equation holds:



(The empty boxes represent fourteen nameless elements, that are the same on each side of the equation and in the same order).

Proof: The associativity of the two operations allows us to shift elements along any row or column independently, as we described above. It is not true in general that one may slide elements in a general array, but middle-four interchange tells us that we can multiply certain 4-tuples of elements in any order (vertical first and the horizontal, and vice-versa). In an array such as above, this condition allows us to slide the elements in either direction as long as there is some "cushion" on the outside. This cushion is, in fact, the outer two elements in the middle-four-shaped 4-tuple that allows us to swap the middle two. Using these facts, we can perform the following operations while fixing the border:

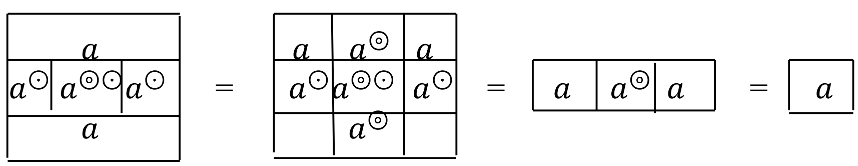




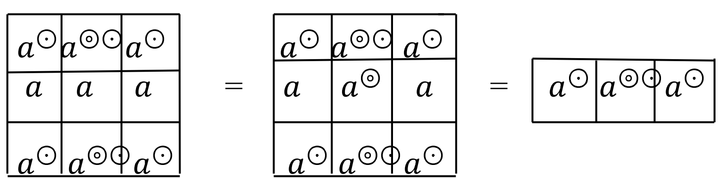
and the result is proved. It is of interest to note that, as long as we have that cushion, we can permute elements in any 2×2 array, as demonstrated by the first eight steps above.

Lemma 3.1 Let S be a double inverse semigroup. Then the inverse operations of S commute. That is $a^{\circledcirc\odot} = a^{\odot\circledcirc}$ for all $a \in S$.

Proof: To prove this result, we first note that



and



These two equations then imply that the vertical inverse of a is $a^{\odot}a^{\circledcirc\odot}a^{\odot}$ or that $a^{\odot} = a^{\odot}a^{\circledcirc\odot}a^{\odot}$

In a similar manner we can calculate the above statements with a replaced by $a^{\circledcirc}$ to see that the vertical inverse of $a^{\circledcirc}$ is $a^{\circledcirc\odot}\, a^{\odot}a^{\circledcirc\odot}$ or that $a^{\circledcirc} = a^{\circledcirc\odot}\, a^{\odot}a^{\circledcirc\odot}$.

Combining these two equations, we conclude that the horizontal inverse of $a^{\odot}$ is $a^{\circledcirc\odot}$ or $a^{\odot\circledcirc} = a^{\circledcirc\odot}$.

We can finally prove the following theorem, also due to Kock:

Theorem 3.2 *Every double inverse semigroup D is commutative.*

Proof: Let $a, b \in D$. Because inverses in either direction are unique, it suffices to show that $a^{\odot}b^{\odot} = b^{\odot}a^{\odot}$. We begin the proof by establishing the following facts :

=

a			b		
$a^{\odot}$	$b^{\odot}$	$a^{\odot\circledcirc}$	$b^{\odot\circledcirc}$	$a^{\odot}$	$b^{\odot}$
a			b		

=

a	b	$b^{\circledcirc}$	$a^{\circledcirc}$	a	b
$a^{\odot}$	$b^{\odot}$	$a^{\odot\circledcirc}$	$b^{\odot\circledcirc}$	$a^{\odot}$	$b^{\odot}$
a		$b^{\circledcirc}$	$a^{\circledcirc}$	a	b

=

a	b	$b^{\circledcirc}$	$a^{\circledcirc}$	a	b
$a^{\odot}$	$b^{\odot}$	$a^{\odot\circledcirc}$	$b^{\odot\circledcirc}$	$a^{\odot}$	$b^{\odot}$
a	b	$b^{\circledcirc}$	$a^{\circledcirc}$	a	b
$a^{\odot}$	$b^{\odot}$	$b^{\odot\circledcirc}$	$a^{\odot\circledcirc}$	$a^{\odot}$	$b^{\odot}$
a	b	$b^{\circledcirc}$	$a^{\circledcirc}$	a	b

=

a	b	$b^{\circledcirc}$	$a^{\circledcirc}$	a	b
$a^{\odot}$	$b^{\odot}$	$b^{\odot\circledcirc}$	$a^{\odot\circledcirc}$	$a^{\odot}$	$b^{\odot}$
a	b	$b^{\circledcirc}$	$a^{\circledcirc}$	a	b
$a^{\odot}$	$b^{\odot}$	$b^{\odot\circledcirc}$	$a^{\odot\circledcirc}$	$a^{\odot}$	$b^{\odot}$
a	b	$b^{\circledcirc}$	$a^{\circledcirc}$	a	b

=

a	b	$a^{\circledcirc}$	$b^{\circledcirc}$	a	b

=

a	b

The first step is to recognize that the horizontal inverse of ab is $(ab)^{\circledcirc} = b^{\circledcirc}a^{\circledcirc}$ and rewrite this six-fold horizontal product. We then add the bottom two rows by trading $a^{\odot}$, $b^{\odot}$, … with their respective conjugations with their vertical inverses. We then swap the $a^{\odot\circledcirc}$ and the $b^{\odot\circledcirc}$, which is justified by Theorem3.1 since we have a 4x4 subrectangle of products. We then recognise that the columns all now collapse to a single element and we can evaluate the remaining six-fold horizontal product.
Similarly, one calculates that which is justified by the Theorem 3.1, since we have a 4x4 subrectangle of products. We then recognize that the columns all now collapse to a single element and we can evaluate the remaining six-fold horizontal product.
Similarly, one calculates that

$a^{\odot}$	$b^{\odot}$	$a^{\circledcirc\odot}$	$b^{\circledcirc\odot}$	$a^{\odot}$	$b^{\odot}$
a			b		
$a^{\odot}$	$b^{\odot}$	$a^{\circledcirc\odot}$	$b^{\circledcirc\odot}$	$a^{\odot}$	$b^{\odot}$

=

$a^{\odot}$	$b^{\odot}$	$a^{\circledcirc\odot}$	$b^{\circledcirc\odot}$	$a^{\odot}$	$b^{\odot}$

These two equations together, then, imply that the vertical inverse of ab is $a^{\odot}b^{\odot}a^{\circledcirc\odot}b^{\odot\circledcirc}a^{\odot}b^{\odot}$ or that (note that $a^{\odot\circledcirc} = a^{\circledcirc\odot}$)

$$a^{\odot}b^{\odot}a^{\circledcirc\odot}b^{\odot\circledcirc}a^{\odot}b^{\odot} = a^{\odot}b^{\odot}$$

If we replace each argument above with its horizontal inverse and do the calculations again, we find that

$$a^{\circledcirc\odot}b^{\odot\circledcirc}a^{\odot}b^{\odot}a^{\circledcirc\odot}b^{\odot\circledcirc} = a^{\circledcirc\odot}b^{\odot\circledcirc}$$

These two equations imply that the horizontal inverse of $a^{\odot}b^{\odot}$ is $a^{\circledcirc\odot}b^{\circledcirc\odot}$. However $(b^{\odot}a^{\odot})^{\circledcirc} = a^{\odot\circledcirc}b^{\odot\circledcirc} = a^{\circledcirc\odot}b^{\circledcirc\odot}$. By uniqueness of inverses, then, $a^{\odot}b^{\odot} = a^{\odot}b^{\odot}$ and we are done.

4. Conclusion

This paper first introduced the reader to the notion of inverse and double semigroups. We then explored the relationship of inverse and regular semigroup and the double inverse semigroup as a special class of inverse semigroups. Then explored some known results about the commutativity in double semigroups and introduced the notion of a double category and showed that a known construction of inverse semigroups with zero from categories, when generalized to constructing double inverse semigroups from double categories. Using these correspondences, we concluded that double inverse semigroups are exactly commutative inverse semigroups and that they are closely related with regular semigroups, so they are as the core semigroups and one of the most studied classes of semigroups.

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