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Conference paper

EVALUATION OF THE COEFFICIENT FURIE IN THE CLASS OF THE FUNCTION WITH CONTINUAL MODUL LIMITED

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Abstract

In this work paper is presented the nature of the function which approximately according n tending to infinity, is the above limit of the Fourier coefficient a_n and b_n, in the set of the functions with continual modul limited with number M, its. $\omega(f,\delta) = \sup\{f(x+h) - f(x); |h| \leq \delta\} \leq M$, M-constant, and this class of the functions we note with $MH[\delta]$.

Here is applied a way where is constructed the function which limited in the above side the Fourier coefficient of this class of the functions. On the other side is presented the different forms of this function with the same conditions and satisfy the result of the theorem. The result in this paper is based in the class of the function limited with number one for set of the function note with $H[\delta]$, but in the same way, this result may to generalized and for the class of the function with limited modul of continue.

Keywords: *approximation, coefficient furie, continual module, class, function*

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1. Introduction

In class H$[\delta]$ – exactly we calculate the uper limit of coefficient furie

Sign in:

$$A_n(\delta) = \{\sup a_n(f); f \in [\delta]\}$$

$$B_n(\delta) = \{\sup b_n(f); f \in [\delta]\}$$

Where a_n, and b_n are the coefficient s furie and for every $\delta > 0$, it.s for function f we have $\omega(f,\delta) \leq 1$ its.

$$\omega(f,\delta) = sup_{|x-y|\leq\delta} |f(x) - f(y)| \leq 1.$$

2. Main results

In the following we (see Lesbg in [2]) conclude that $A_n\left(\frac{\pi}{n}\right)$ and $B_n\left(\frac{\pi}{n}\right)$ approximated with $\frac{2}{\pi}$.

Teoreme 1 ([2]): If n- is a natural fixed number ,then :

$A_n(\delta) = B_n(\delta) = \omega(n\delta)$, where (1)

$$\omega(\delta)=\begin{cases}\frac{2\cos\frac{\delta}{2}<\frac{\pi}{\delta}>}{\pi\sin\frac{\delta}{2}}, & 0<\delta\leq\frac{2\pi}{3}\\ \frac{2}{\pi}, & \delta>\frac{2\pi}{3}\end{cases} \qquad (2)$$

Proof: From identity

$$\int_{-x}^{x} f(x)\cos nx\,dx=\int_{-x}^{x} f(x-\frac{\pi}{2n})\sin nx\,sx,$$

because for x=(t-$\frac{\pi}{2n}$) , we have

$$\cos n(t-\frac{\pi}{2n})=\cos(nt-\frac{\pi}{2})=\sin nt$$

and because f is the 2π – periodic function also and sinnx , then integral in interval $\left[-\pi+\frac{\pi}{2n},\pi+\frac{\pi}{2n}\right]$ is equal with integral in $[-\pi,\pi]$.

Consider that f(x) $\in H[\delta]$ we have f($x-\frac{\pi}{2n}$) $\in H[\delta]$, because from $|x-y|\leq\delta$ and $\left|x-\frac{\pi}{2n}-y+\frac{\pi}{2n}\right|=|x-y|\leq\delta$, its. ,

$$A_n(\delta)=B_n(\delta). \qquad (3)$$

If function $f\in H[\delta]$, then :

$$f_1(x)=\frac{1}{n}\sum_{k=0}^{n-1} f(x=\frac{2k\pi}{n})\in H[\delta]$$

To consider that $|x_1 - x_2|\leq\delta$, we have

$$|f_1(x_1-f_1(x_2)|=\frac{1}{n}\left|\sum_{k=0}^{k=n-1}((f_1(x_1+\frac{2k\pi}{n})-f_1(x_2+\frac{2k\pi}{n}))\right|\leq$$

$$\frac{1}{n}\sum_{k=0}^{n-1}\left|f_1(x_1+\frac{2k\pi}{n})-f_1(x_2+\frac{2k\pi}{n})\right|\leq\frac{1}{n}\sum_{k=0}^{n-1}1=1 \qquad ,$$

because

$$\sup_{|x_1-x_2\leq\delta|}|f_1(x_1-f_1(x_2)| \quad \leq 1,$$ its. $\omega(f_1,\delta)\leq 1$, thus $f_1\in H(\delta)$.

In continue

$$f_1(x_1+\frac{2\pi}{n})=\frac{1}{n}\sum_{k=0}^{k=n-1}(f\quad(x_1+\frac{2k\pi}{n}+\frac{2\pi}{n})=\frac{1}{n}\sum_{k=0}^{k=n-1}(f\quad(x_1+\frac{2k\pi}{n})=f_1(x),$$

because f(x) is the 2π- periodic function , from that for k=0 and k=n we get $f\left(x+\frac{2k\pi}{n}\right)=f(x)$, its. $f_1(x)$ is $\frac{2\pi}{n}$ periodic.

To consider

$$\frac{1}{n}\int_{-\pi}^{\pi} f_1(x)\sin nx\,dx=\frac{1}{n\pi}\int_{-}^{\pi}(\sum_{k=0}^{n-1} f(x+\frac{2k\pi}{n}))\sin nx dx=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin x\,dx,$$

the case for evaluation of $B_n(\delta)$ we limited with function $\frac{2\pi}{n}$ – periodic, because even the function f(x)sinnx is $\frac{2\pi}{n}$- periodic, its.

$$b_n(f)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx=\frac{n}{\pi}\int_{-\frac{\pi}{n}}^{\frac{\pi}{n}}f(x)\sin nx\,dx,$$

and if for x we get $\frac{x}{n}$, we have

$$b_n(f)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx=\frac{n}{\pi}\int_{-\frac{\pi}{n}}^{\frac{\pi}{n}}f(x)\sin nx\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}f\left(\frac{x}{n}\right)\sin x\,dx. \qquad (4)$$

For $|x_1 - x_2|\le n\delta$, we obtain $\left|\frac{x_1}{n}-\frac{x_2}{n}\right|\le\delta$, from thus we conclude that the function $f(\frac{x}{n})$ belongs the class $H[n\delta]$.

Then to consider (4) for $B_n(\delta)$, we have

$$B_n(\delta)=\sup_{f\in H[\delta]}\frac{1}{\pi}\int_{=\pi}^{\pi}f(x)\sin nx\,dx=\sup_{f\in H[n\delta]}\frac{1}{\pi}\int_{=\pi}^{\pi}f(x)\sin x\,dx$$

its. $B_n(\delta)=B_1(n\delta)$

From the above relation we can see that the problem is to find the supremum of the first coefficient furie $b_1(f)$. Then for $B_1(\delta)$ is enough to get even function $f\in H[\delta\}$, because $f_2(x)=\frac{f(x)-f(-x)}{2}$ is from $H[\delta\}$, and in continue we can see that

$$f_2(-x)=\frac{f(-x)-f(x)}{2}=-f_2(x) \quad \text{than} \quad \frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin x\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}f_2(x)\sin x\,dx$$

For coefficient $b_1(f)$, we get

$$b_1(f)=\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{3\pi}f(x)\sin x\,dx=\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}f(x)\sin x\,dx-\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}f(x+\pi)\sin x\,dx,$$

because

$$\int_{-\frac{\pi}{2}}^{3\pi}f(x)\sin x\,dx=-\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}f(x+\pi)\sin x\,dx$$

where $x=t+\pi$ and $\sin(t+\pi)=-\sin t$.

From above relation, we get

$$\sup_{f\in H[\delta]}b_1(f)\le\sup_{f\in H[\delta]}\frac{1}{\pi}\left|\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}f(x)\sin x\,dx\right|+\left|\sup_{f\in H[n\delta]}\frac{1}{\pi}\int_{=\frac{\pi}{2}}^{\frac{\pi}{2}}f(x+\pi)\sin x\,dx\right| \qquad (5)$$

and for $f(x)\in H[\delta\}$ also $f(x+\pi)\in H[\delta\}$.

Example 1. (see [1]) If [x] satisfy, {x}=x-[x],

<x>=min{{x}, 1-{x}} and

$$g(x)=\begin{cases}\cos x, \text{per } |x|\le \frac{\pi}{2}\\ 0, \text{per } |x|\ge \frac{\pi}{2}\end{cases},$$

then

$$\sup_{f\in H[\delta]}\left|\int_{-\infty}^{\infty} f(x)\sin x\,dg(x)\right| = \begin{cases}\frac{\cos(\frac{\delta}{2}<\frac{\delta}{2}>)}{\sin\frac{\delta}{2}} \text{ per } 0<\delta\le\frac{2\pi}{3}\\ 1, \text{ per } \delta\ge\frac{2\pi}{3}\end{cases}.$$

From (5) and the example given above, we obtain

$$B_1(\delta)\le\varphi(\delta). \quad (6)$$

In other side, if $h(x)$ is 2π periodic function, defined in $[-\frac{\pi}{2},\frac{3\pi}{2}]$ with

$$h(x)=\begin{cases} g(x), \text{per } x\in[-\frac{\pi}{2},\frac{\pi}{2}]\\ -g(x-\pi), \text{per } x\in[\frac{\pi}{2},\frac{3\pi}{2}]\end{cases}.$$

Where $g(x)=\left[\frac{x-\theta}{\delta}\right]-\left[\frac{-\theta}{\delta}\right], 0\le\delta\le\pi$ and θ the number which satisfy

$$\left\{\frac{\theta}{\delta}-\frac{\pi}{2\delta}\right\}-\left\{\frac{\theta}{\delta}+\frac{\pi}{2\delta}\right\}=0, \left\{\frac{\theta}{\delta}-\frac{\pi}{2\delta}\right\}+\left\{\frac{\theta}{\delta}-\frac{\pi}{2\delta}\right\}=1-<\frac{\pi}{\delta}>,$$

then

$$\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} h(x)\sin x\,dx=\varphi(\delta) \qquad (7)$$

To consider (6) and (7) we can conclude the prooth of the teorem.

Corollary 1. Function φ may to present and with the form

$$\varphi(\delta)=\begin{cases}\frac{2\sin(s+\frac{1}{2})\delta}{\sin\frac{\delta}{2}}, \frac{2\pi}{4s+3}\le\delta\le\frac{2\pi}{4s+2}, s\in N\\ \frac{2\sin\delta}{\sin\frac{\delta}{2}}, \frac{2\pi}{4s+1}\le\delta\le\frac{2\pi}{4s-1}, s\in N\\ \frac{2}{\pi}, \delta\ge\frac{2\pi}{3}\end{cases}. \qquad (8)$$

We can see that, if $\frac{2\pi}{4s+3}\le\delta\le\frac{2\pi}{4s+2}$, then $2s+1\le\frac{\pi}{6}\le 2s+\frac{3}{2}, s\in N$, from that

$$\left[\frac{\pi}{\delta}\right]=2s+1, \left\{\frac{\pi}{\delta}\right\}=\frac{\pi}{\delta}-2s-1\le\frac{1}{2}$$

because $\frac{\pi}{\delta}-(2s+1)\le 2s+\frac{3}{2}-2s-1=\frac{1}{2}$, thus

$$1-\left\{\frac{\pi}{\delta}\right\}=2s+2-\frac{\pi}{\delta}\ge\frac{1}{2}, \qquad 2s+2-2s-1=1>\frac{1}{2}.$$

$$\cos\frac{\delta}{2}<\frac{\pi}{\delta}>=cos(\frac{\delta}{2}\left\{\frac{\pi}{\delta}\right\})=cos(\frac{\delta}{2}(\frac{\pi}{\delta}-(2s+1)=\sin(\delta(2s+1)). \qquad (9)$$

For

$\frac{2\pi}{4s+2}\le\delta\le\frac{2\pi}{4s+1}$, we get $\frac{4s+1}{2}\le\frac{\pi}{\delta}\le\frac{4s+2}{2}$, then $2s+\frac{1}{2}\le\frac{\pi}{\delta}\le 2s+1$, its.

$$\left[\frac{\pi}{\delta}\right] = 2s, \left\{\frac{\pi}{\delta}\right\} = \frac{\pi}{\delta} - 2s \geq 2s + \frac{1}{2} - 2s = \frac{1}{2},$$

and

$$1 - \left\{\frac{\pi}{\delta}\right\} = 1 - \frac{\pi}{\delta} + 2s \leq 2s + 1 - 2s - \frac{1}{2} = \frac{1}{2}$$

Its.

$$< \frac{\pi}{\delta} \geq \min(\left\{\frac{\pi}{\delta}\right\}, 1 - \left\{\frac{\pi}{\delta}\right\}) = 2s + 1 - \frac{\pi}{\delta} \geq \frac{1}{2}$$

and

$$\cos\frac{\delta}{2} < \frac{\pi}{\delta} >= \sin((s + \tfrac{1}{2})\delta). \qquad (10)$$

For $\frac{2\pi}{4s+1} \leq \delta \leq \frac{2\pi}{4s}$, we have $2s \leq \frac{\pi}{\delta} \leq 2s + \frac{1}{2}$,

From that

$\left[\frac{\pi}{\delta}\right] = 2s, \left\{\frac{\pi}{\delta}\right\} = \frac{\pi}{\delta} - 2s \leq 2s + \frac{1}{2} - 2s = \frac{1}{2}$,

$$1 - \left\{\frac{\pi}{\delta}\right\} = 1 - \frac{\pi}{\delta} + 2s \geq 2s + 1 - 2s - \frac{1}{2} = \frac{1}{2}$$

its. $< \frac{\pi}{\delta} > = \frac{\pi}{\delta} - 2s$

then

$$\cos\frac{\delta}{2} < \frac{\pi}{\delta} >= \cos\left(\frac{\pi}{2} - \delta s\right) = \sin(s\delta). \qquad (11)$$

For

$$\frac{2\pi}{4s} \leq \delta \leq \frac{2\pi}{4s-1} \qquad 2s - \frac{1}{2} \leq \frac{\pi}{\delta} \leq 2s,$$

$$\left[\frac{\pi}{\delta}\right] = 2s - 1, \left\{\frac{\pi}{\delta}\right\} = \frac{\pi}{\delta} - 2s + 1 \geq 2s - \frac{1}{2} - 2s + 1 = \frac{1}{2},$$

$$1 - \left\{\frac{\pi}{\delta}\right\} = 2s - \frac{\pi}{\delta} \leq \frac{1}{2}, < \frac{\pi}{\delta} > 2s - \frac{\pi}{\delta}$$

then

$$\cos\frac{\delta}{2} < \frac{\pi}{\delta} >= \sin(s\delta). \qquad (12)$$

Consider (9), (10), (11), and (12) we can conclude the proof of (8).

In continue we present the aproximation of $A_n(\delta)$, and $B_n(\delta)$ where $\delta \to \infty$.

Theoreme 2. If n naturale number, then

$$A_n(\delta) = B(\delta) \approx \frac{4}{n\pi\delta}, \quad \delta \to \infty \ .$$

Conclusion

For $\delta > 0$, we can get the natural number n satisfy $0 < \frac{\pi}{n\delta} \leq \frac{1}{2}$, then

$\frac{n\delta}{2} < \frac{\pi}{n\delta} > \to \infty$, where $\delta \to \infty$.

Consider the theorem 1, we have

$$\lim_{\delta \to 0} \frac{A_n(\delta)}{\frac{4}{n\pi\delta}} = \lim_{\delta \to 0} \frac{\frac{2cos\frac{n\delta}{2} < \frac{\pi}{n\delta} >}{\pi sin\frac{n\delta}{2}}}{\frac{4}{n\pi\delta}} = \lim_{\delta \to 0} \frac{cos\frac{n\delta}{2}}{\frac{2}{n\delta} sin\frac{n\delta}{2}} = 1,$$

In the similary way we may to calculate and for $B_n(\delta)$,which the proof of the theorem complet'

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