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Conference paper

NUMERICAL SOLUTION AND CHAOTIC DYNAMIC OF FRACTIONAL-ORDER LORENZ AND CHEN SYSTEM

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Abstract

The control of the chaotic dynamical systems of fractional-order has been utilized in various applications in mechanics, physics, control theory, and other engineering areas. This paper is devoted to a numerical and theoretical analysis of nonlinear fractional-order systems, namely the chaotic Lorenz and Chen system, with different topological structure of attractors. We use the method of Multistep Fractional Differential Transform (FMDTM) as an analytical and numerical method for solving a wide variety of fractional differential equations, which will increase the interval of convergence for the series solution. In this case, it turns out that the Chen system is more sensitive to initial conditions than the Lorenz system. We use a reliable algorithm, Fractional Multistep Differential Transform method (FMDTM) with Drops to compare the results. Numerically obtained results are analyzed to compare various integration algorithms. The computer simulations demonstrate the reliability and efficiency of the algorithm developed.

Keywords: *Fractional-order differential equation, Lorenz system, Chen system, Fractional Multistep Differential Transform method, numerical results*.

1. Introduction

Fractional calculus is three centuries old. The beauty of this part of the science is that fractional derivatives (integrals) are not a local (point) property (quantity). The idea of fractional calculus has been known since the regular calculus, with the first reference probably being associated with Leibniz and L'Hospital in 1965 where half-order derivative was mentioned.

At present, the number of applications of fractional calculus rapidly grows, these mathematical phenomena allow us to describe and model a real object more accurately than the classical "integer" methods.

The fractional-order calculus plays an important role in physics, thermodynamics, electrical circuit theory, mechatronic systems, signal processing, chemical mixing, chaos theory, etc. Abel was the first who wrote a fractional equation for solving the tautochrone problem. In this paper we focus on the nonlinear fractional systems of the form:

$$D_i^{q_i} x_i(t) = f_i(x_1(t), x_2(t), \dots, x_n(t), t)$$

$$x_i(0) = c_i, i = 1,2, \dots, n \qquad (1)$$

where the c_i are the initial conditions, or in its vector representation:

$$D^q x = f(x) \tag{2}$$

where $q = [q_1, q_2, \dots, q_n]^\top$ for $0 < q_i < 2, (i = 1,2, \dots, n)$ and $x \in \mathbb{R}^n$.

The equilibrium points of the system (2) are calculated via solving following equation: $f(x) = 0$. Those kind of systems are very interesting to engineers, physicists, and mathematicians, because most real physical systems are inherently nonlinear in nature, especially we will discuss about a well-known nonlinear system which exhibits chaos, the Lorenz system, and its application. The exact solution of those kinds of systems is not possible to be found, so we solve them numerically by using a different kind of methods. We use (FMDTM), an analytic and numerical method, currently used, as a technique for analytic calculating the power series of the solution.

This paper is organized as follows. Section 2 is a brief introduction of fractional calculus. Section 3 is on the fractional-order system of Lorenz containing the conditions of its stability. Section 4 is on the numerical method (FMDTM), the part of the theory and numerical results.

2. Fractional Calculus

Here, we should mention the basics of the fractional calculus, fractional integral and derivative. The fractional integral of order q for function $f(t)$ can be expressed as follows:

$$I_t^q f(t) = D_t^{-q} = \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f(\tau) d\tau \tag{3}$$

Caputo definition of the fractional derivative of order q is:

$$D_t^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t \int_0^t (t-\tau)^{q-1} f^{(n)}(\tau) d\tau \tag{4}$$

It is chosen to use Caputo fractional derivative because it allows initial and boundary conditions to be included in the formulation of the problem, even that for homogeneous initial conditions, these two operators coincide.

Lorenz system of fractional order,

$$D_t^{q_1} x(t) = a(y(t) - x(t))$$

$$D_t^{q_2} y(t) = cx(t) - x(t)z(t) - y(t)$$

$$D_t^{q_3} z(t) = x(t)y(t) - bz(t) \tag{5}$$

Chen system of fractional order,

$$D_t^{q_1} x(t) = a(y(t) - x(t))$$

$$D_t^{q_2} y(t) = (c-a)x(t) - x(t)z(t) + cy(t)$$

$$D_t^{q_3} z(t) = x(t)y(t) - bz(t) \tag{6}$$

$x(0) = c_1, y(0) = c_2, z(0) = c_3$, where $D_t^{q_i}$ are Caputo fractional derivative for $i = 1, 2, 3$, a, b, c are real parameters and $q_i \in (0,1]$ the fractional order. In the continuous work, we will discuss the system where $q_1 = q_2 = q_3 = q$.

Stability as an extremely important property of dynamical systems can be investigated in various domains, one of them is the frequency domain, but if we deal with incommensurate fractional-order systems, it is important to bear in mind that $P(s^\alpha), \alpha \in \mathbb{R}$ is a multivalued function of $s^\alpha, \alpha = u/v$, the domain of which can be viewed as a Riemann surface with finite number of Riemann sheets v, where the origin is a branch point and the branch cut is assumed as $\mathbb{R}^-$. It is fact that in multivalued functions only the first Riemann sheet has it physical significance.

Lorenz system has three equilibria, where one is obviously in origin $E_1(0;0;0)$ and the other two are: $E_2(\sqrt{(bc-c)}; \sqrt{(bc-c)}; c-1), E_3(-\sqrt{(bc-c)}; -\sqrt{(bc-c)}; c-1)$. The Jacobian matrix of Lorenz's system (6) at the equilibrium point $E^*(x^*, y^*, z^*)$ is given by:

(7)
$$J = \begin{bmatrix} -a & a & 0 \\ c-z^* & -1 & -z^* \\ y^* & x^* & -b \end{bmatrix}$$

We will investigate the fractional -order Lorenz's system by changing the initial conditions and showing their numerical methods.

The equilibrium points of the system of Chen with the above parameters are: $E_1(0;0;0)$, E_2 (7.9373;7.9373;21) and E_3 (−7.9373;−7.9373;21), with Jacobian matrix:

(8)
$$J = \begin{bmatrix} -a & a & 0 \\ c-a-z^* & c & -x^* \\ y^* & x^* & -b \end{bmatrix}$$

3. Numerical methods

3.1 Fractional Multistep Differential Transformation Method

An important part of the paper is to present approximate analytical solutions for the Lorenz system with fractional-order (5). The fractional multistep differential transform method (FMDTM) is a numerical method based on the Taylor series expansion which constructs an analytical solution in the form of a polynomial. The traditional high order Taylor series method requires symbolic computation. However, the differential transform method obtains a polynomial series solution using an iterative procedure. Firstly, expand the analytic function $f(t)$ in terms of fractional power series as follows:

(9)
$$f(t) = \sum_{k=0}^{\infty} F(k)(t-t_0)^{kq}$$

Where $0 < q \leq 1$ is the order of fractional derivative and $F(k)$ is the fractional differential transform of $f(t)$, is given as

(10)
$$F(k) = \frac{1}{\Gamma(qk+1)}\left[\left(D_{t_0}^q\right)^k (f(t_0))\right]$$

where $\left(D_{t_0}^q\right)^k = D_{t_0}^q \cdot D_{t_0}^q \cdot \ldots \cdot D_{t_0}^q$ the k-times differential Caputo fractional derivative. In our application, we will approximate the function $f(t)$ by the finite series, so the finite form of (9):

$$f(t) = \sum_{k=0}^{N} F(k)(t - t_0)^{kq}$$

We apply the (FMDTM) and the differential transformation for the Lorenz system is:

$$\frac{\Gamma(q(k+1)+1)}{\Gamma(qk+1)} X(k+1) = a(Y[k] - X[k])$$

$$\frac{\Gamma(q(k+1)+1)}{\Gamma(qk+1)} Y(k+1) = cX(k) - \sum_{l=0}^{k} X(l)\, Z(k-l) - Y(k)$$

$$\frac{\Gamma(q(k+1)+1)}{\Gamma(qk+1)} Z(k+1) = \sum_{l=0}^{k} X(l)\, Y(k-l) - bZ(k)$$

where $X(0) = c_1, Y(0) = c_2, Z(0) = c_3$.

If we consider the fractional order series $f(t) = x(t) = \sum_{m=0}^{N} F(m)(t - t_0)^{\frac{k}{\alpha}}$, then the transformed method has the following form:

$$\frac{\Gamma\left(\frac{k}{\alpha}+1\right)}{\Gamma\left(q+1+\frac{k}{\alpha}\right)} X(k+1) = \ a(Y[k] - X[k])$$

$$\frac{\Gamma\left(\frac{k}{\alpha}+1\right)}{\Gamma\left(q+1+\frac{k}{\alpha}\right)} Y(k+1) = cX(k) - \sum_{l=0}^{k} X(l)\, Z(k-l) - Y(k)$$

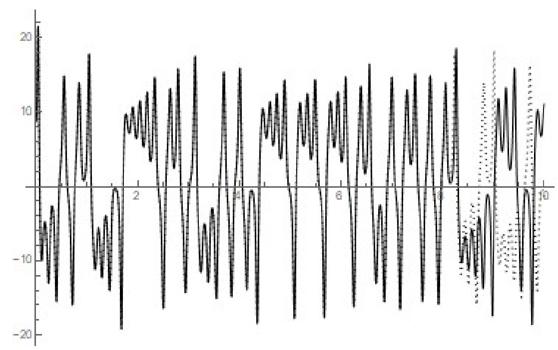
$$\frac{\Gamma\left(\frac{k}{\alpha}+1\right)}{\Gamma\left(q+1+\frac{k}{\alpha}\right)} Z(k+1) = \sum_{l=0}^{k} X(l)\, Y(k-l) - bZ(k)$$

Numerical results for a system (5) are taken using the 10-order solutions with the inverse transformation. We will name them like (FMDTM) and (FMDTM1). The same transformations we use for the second system, fractional-order Chen system.

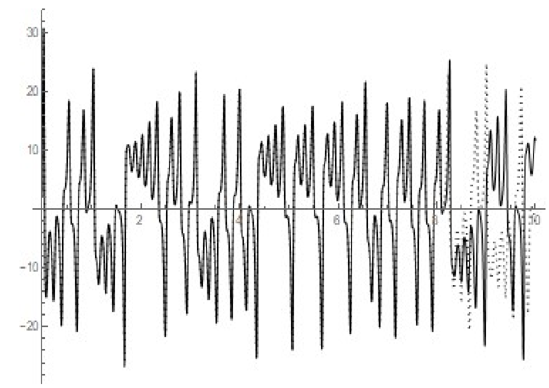
4. Numerical results

We will consider the values$a = 10,\ b = \frac{8}{3}, c = 28,$ for the Lorenz system and $a = 35$, $b = 3$, and $c = 28$ for the Chen system, usual cases, but investigated in different ways. We took the risk to show how the (FMDTM) and (FMDTM1) methods will act, on the initial conditions $x(0) = 10,\ y(0) = 1$ and $z(0) = 0.1$ for an fractional-order from interval$(0,1]$, which is $q = 0.8$, even that the conditions to have chaos is $q > 0.994$. We observe the approximated solutions of the both systems (5) and (6), with above-mentioned parameters, using two numerical methods (FMDTM) and (FMDTM1). The purpose is to compare the results for each fractional-order and showing that in which conditions they are in good agreement with each other by plotting them, using Mathematica 11.0 Package.

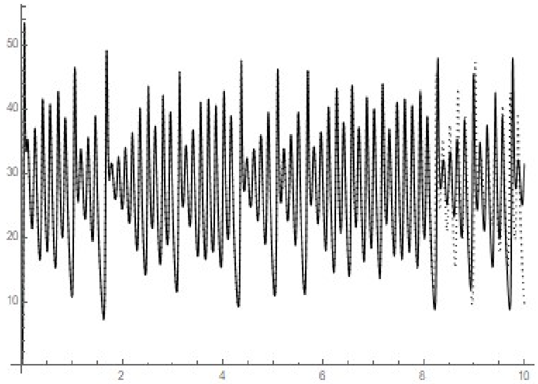
We will share the results on time interval $[0,10]$, with step size $h = 0.001$, for $x(t), y(t)$ and $z(t)$.



x(t), with h=0.001, x(0)=10 and t∈[0,10].

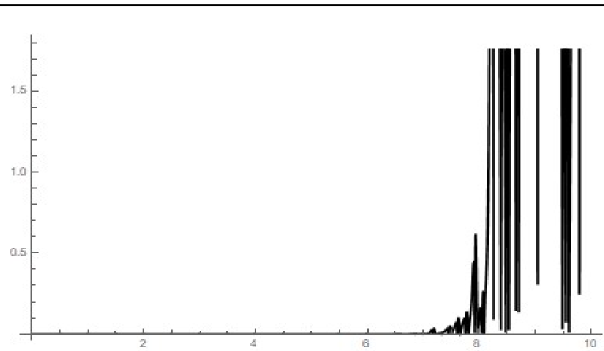


y(t), with h=0.001, y(0)=1 and t∈[0,10]

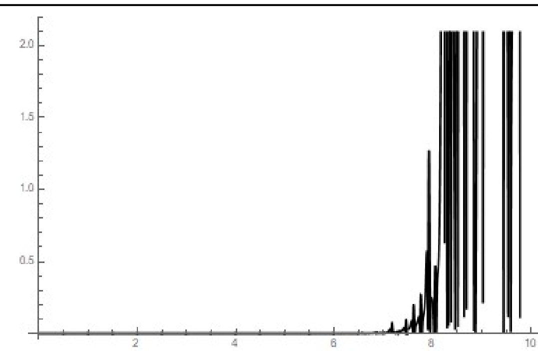


y(t), with h=0.001, y(0)=1 and t∈[0,10]

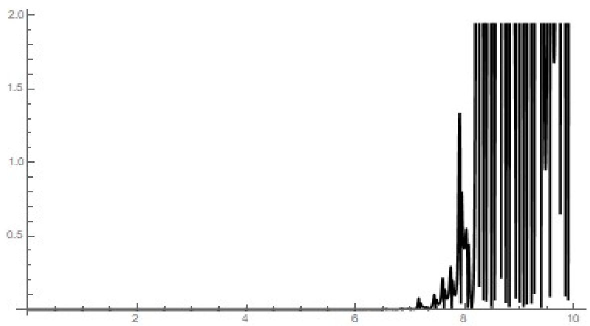
Figure 1. The Lorenz system's time-series with standard attractor *a* = 10, *b* =8/3, and *c* = 28, initial conditions, *x*(0) = 10, *y*(0) =1 and *z*(0) = 0.1, same fractional-order *v*=0.8 and same step-size *h*=0.001.



ABS[FMDTM-FMDTM1] for x(t)

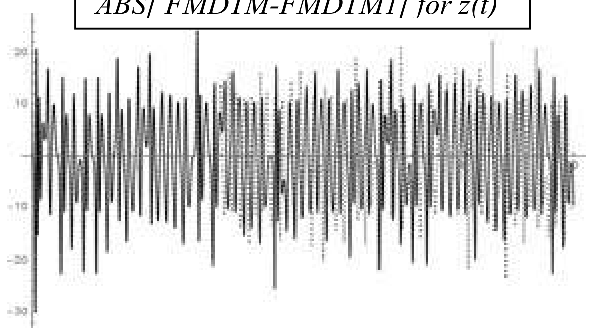


ABS[FMDTM-FMDTM1] for y(t)

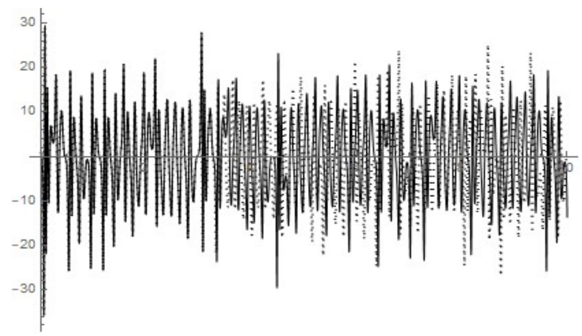


ABS[FMDTM-FMDTM1] for z(t)

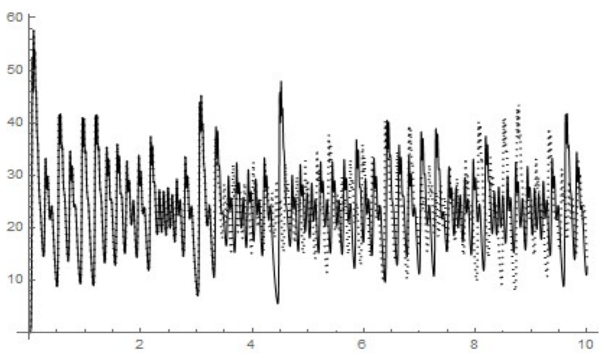
Figure 2. Absolute difference for the time series x(t), y(t) and z(t) of the fractional-order Lorenz system , for v = 0.8. The step size is h = 0.001 and $t \in [0,10]$.



x(t), with h=0.001, x(0)=10 and t∈[0,10].

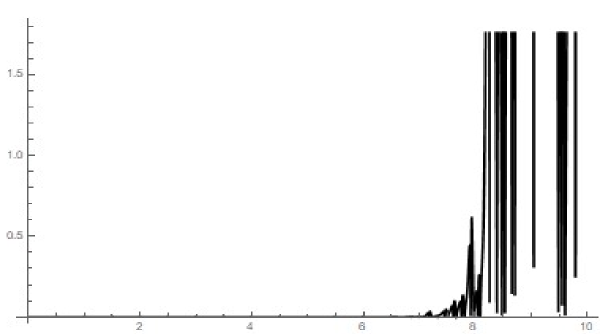


y(t), with h=0.001, y(0)=1 and t∈[0,10]

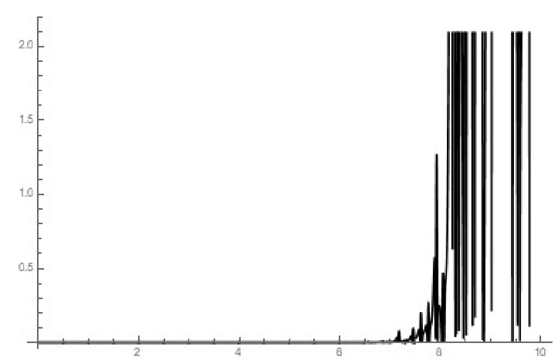


y(t), with h=0.001, y(0)=1 and t∈[0,10]

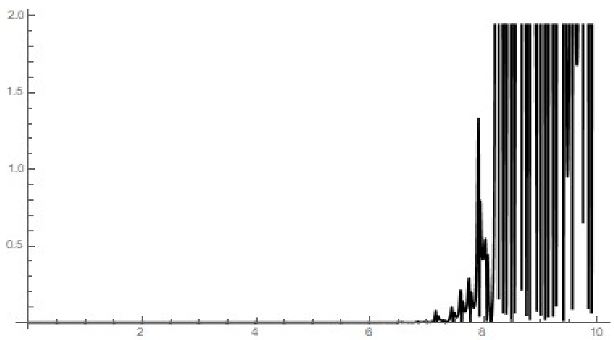
Figure 3. Chen system's time-series with standard attractor a = 35, b =3, and c = 28, initial conditions, x(0) = 10, y(0) =1 and z(0) = 0.1, same fractional-order v=0.8 and same step-size h=0.001.



ABS[FMDTM-FMDTM1] for x(t)



ABS[FMDTM-FMDTM1] for y(t)



ABS[FMDTM-FMDTM1] for z(t)

Figure 4. Absolute difference for the time series x(t), y(t) and z(t) of the fractional-order Chen system , for $v = 0.8$. The step size is $h = 0.001$ and $t \in [0,10]$.

5. Conclusions

In this paper, two different numerical approximation schemes (FMDTM and FMDTM1) have been applied to find the time-series solutions of the fractional-order Lorenz and Chen system. We have aimed to quantify the distinction between the integration methods by depicting the time series of the absolute difference for the same system parameters and initial conditions, with the order of the fractional derivative v=0.8. The results show that (FMDTM) and (FMDTM1) methods are in excellent agreement, but they start to differ after the length t=6 of interval (0,10).

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