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A NOTE ON A RESULT ABOUT SKEW BINORMAL OPERATORS

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Abstract

In this paper we will give some counterexamples that will show that class of binormal operators and skew binormal operators are independent. Even more, that the class of $n-$binormal operators and skew $n-$binormal operators are independent. This would be used to achieve our aim to shown that a result in [4] is not correct.

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1. Introduction

Throughout this paper H it is a Hilbert space and $B(H)$ is the algebra of all bounded linear operators acting on H. If $T \in B(H)$, then with $T^* \in B(H)$ we denote the adjoint of T. Operator $T \in B(H)$ is binormal if T^*T commute with TT^*, that is if $(T^*T)(TT^*) = (TT^*)(T^*T)$. About class of binormal operators the reader can see [1]. On definition and some properties of skew binormal operators can see [2]. The class of skew $n-$binormal operators was introduced in [4]. The formal definitions of operators mentioned above are given bellow.

Definition 1.1. [1] An operator $T \in B(H)$ is binormal if $(T^*T)(TT^*) = (TT^*)(T^*T)$.

Definition 1.2. [2] An operator T is skew binormal operator if $(T^*TTT^*)T = T(TT^*T^*T)$.

Definition 1.3. [3] An operator T is $n-$binormal operator if satisfy $T^*T^nT^nT^* = T^nT^*T^*T^n$. We denote this class with [nBN].

Definition 1.4. [4] An operator T is skew $n-$binormal if $[T^*T^nT^nT^*]T = T[T^nT^*T^*T^n]$. We denote this class with [$snBN$].

2. Main result

In article [2], in Theorem 2.2. authors claim that if an operator is binormal, then it is a skew-binormal one. In this point, we have some counterexamples that deny this claim. In fact, an operator can be binormal, but doesn't need to be a skew-binormal operator as shown example bellow.

Example 2.1. The first part of this example is in [4]. Let

$$T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}.$$

Then, T is a binormal operator. Indeed, we can calculate to find that

$$T^*TTT^* = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} = TT^*T^*T .$$

But, on other side we find that

$$(T^*TTT^*)T = \begin{bmatrix} 0 & 0 & 2 \\ 4 & 4 & 0 \\ 2 & -2 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 & 2 \\ 2 & 4 & 0 \\ 2 & -4 & 0 \end{bmatrix} = T(TT^*T^*T)$$

which prove that such an operator is not skew binormal one.

It is not the only one operator (matrix) with this property. Every matrix, real or complex, let denote by

$$T = \begin{bmatrix} 0 & 0 & a \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

is binormal, but not skew binormal. Indeed, calculating, with

$$T^* = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ \bar{a} & 0 & 0 \end{bmatrix}$$

we find that

$$T^*TTT^* = \begin{bmatrix} 2|a|^2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2|a|^2 \end{bmatrix} = TT^*T^*T$$

which proves binormality. On the other hand we have

$$(T^*TTT^*)T=\begin{bmatrix}2|a|^2 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 2|a|^2\end{bmatrix}\begin{bmatrix}0 & 0 & a\\ 1 & 1 & 0\\ 1 & -1 & 0\end{bmatrix}=\begin{bmatrix}0 & 0 & 2a|a|^2\\ 4 & 4 & 0\\ 2|a|^2 & -2|a|^2 & 0\end{bmatrix}$$

and

$$T(TT^*T^*T)=\begin{bmatrix}0 & 0 & a\\ 1 & 1 & 0\\ 1 & -1 & 0\end{bmatrix}\begin{bmatrix}2|a|^2 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 2|a|^2\end{bmatrix}=\begin{bmatrix}0 & 0 & 2a|a|^2\\ 2|a|^2 & 4 & 0\\ 2|a|^2 & -4 & 0\end{bmatrix}$$

It is clear that, in general, $(T^*TTT^*)T\neq T(TT^*T^*T)$ which proves that such a matrix is not skew binormal one.

Example 2.2. The first part of this examples is in [4]. An operator can be skew binormal (skew $n-$binormal), but need not to be binormal ($n-$binormal). The matrix

$$T=\begin{bmatrix}0 & 1 & 1\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}$$

is not 2-binormal because

$$T^*T^2T^2T^*=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\neq\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}=T^2T^*T^*T^2,$$

but is it a skew 2-binormal operator as shown this calculating

$$(T^*T^2T^2T^*)T=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 1 & 1\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

and

$$T(T^2T^*T^*T^2)=\begin{bmatrix}0 & 1 & 1\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}.$$

That is $(T^*T^2T^2T^*)T=T(T^2T^*T^*T^2)$.

Taking a family of matrix (real or complex) represented by $T=\begin{bmatrix}0 & 1 & a\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}$, we can easily see that

T is not 2-binormal operator, because

$$T^*T^2T^2T^* = \begin{bmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ \bar{a} & 1 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ \bar{a} & 1 & 0\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & \bar{a}\end{bmatrix}\begin{bmatrix}\bar{a} & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

and

$$T^2T^*T^*T^2 = \begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ \bar{a} & 1 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ \bar{a} & 1 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} = \begin{bmatrix}\bar{a} & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & \bar{a}\end{bmatrix} = \begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}.$$

On the other hand we find that

$$(T^*T^2T^2T^*)T = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 1 & a\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \text{ and}$$

$$T(T^2T^*T^*T^2) = \begin{bmatrix}0 & 1 & a\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$ no matter what the value of a is, which means

skew 2-binormality.

We resume that, the condition at Theorem 2.2. [2] is weak to guarantee the skew binormality, and must be improved.

Our conclusions can be formulated in following propositions.

Proposition 2.3. *The class of binormal operators and skew binormal operators are independent.*

Proposition 2.4. *The class of $n-$binormal operators, [nBN], and skew $n-$binormal operators, [$snBN$], are independent.*

References

[1]. Campbell, S.L, (1972) Linear operators for which T^*T and TT^* commute, *Proc. Am. Math. Soc.*, 341, 74 – 80.

[2]. Meenambika, K., Seshaiah C.V, & Sivamani, N., (2018) Skew binormal operators acting on a Hilbert space, *IOP Conf. Series: Journal of Physics: Conf. Series* 1139, 012056 doi:10.1088/1742-6596/1139/1/102056.

[3]. Panayappan, S., & Sivamani, N., (2012) On $n-$binormal Operators, *Gen. Math. Notes, Vol*. 10, No.2, 1 – 8.

[4]. Rasimi, K., Seferi, Y., Ibraimi, A., & Sadiki, F., (2019). Skew $n-$Binormal Operators, *International Journal of Mathematical Analysis*, *Vol.* 13, No. 11, 513 – 517.