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A NOTE ON A RESULT ABOUT SKEW BINORMAL OPERATORS

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Abstract

In this paper we will give some counterexamples that will show that class of binormal operators and skew binormal operators are independent. Even more, that the class of $n-$ binormal operators and skew $n-$ binormal operators are independent. This would be used to achieve our aim to shown that a result in [2] is not correct.

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1. Introduction

Throughout this paper H it is a Hilbert space and $B(H)$ is the algebra of all bounded linear operators acting on H. If $T \in B(H)$, then with $T^* \in B(H)$ we denote the adjoint of T. Operator $T \in B(H)$ is binormal if T^*T commute with TT^*, that is if $(T^*T)(TT^*) = (TT^*)(T^*T)$. About class of binormal operators the reader can see [1]. On definition and some properties of skew binormal operators can see [2]. The class of skew $n-$binormal operators was introduced in [4]. The formal definitions of operators mentioned above are given bellow.

Definition 1.1. [1] An operator $T \in B(H)$ is binormal if $(T^*T)(TT^*) = (TT^*)(T^*T)$.

Definition 1.2. [2] An operator T is skew binormal operator if $(T^*TTT^*)T = T(TT^*T^*T)$.

Definition 1.3. [3] An operator T is $n-$binormal operator if satisfy $T^*T^nT^nT^* = T^nT^*T^*T^n$. We denote this class with [nBN].

Definition 1.4. [4] An operator T is skew $n-$binormal if $[T^*T^nT^nT^*]T = T[T^nT^*T^*T^n]$. We denote this class with [$snBN$].

2. Main result

In article [2], in Theorem 2.2. authors claim that if an operator is binormal, then it is a skew-binormal one. In this point, we have some counterexamples that deny this claim. In fact, an operator can be binormal, but doesn't need to be a skew-binormal operator as shown example bellow.

Example 2.1. The first part of this example is in [4]. Let

$$T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}.$$

Then, T is a binormal operator. Indeed, we can calculate to find that

$$T^*TTT^* = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} = TT^*T^*T\ .$$

But, on other side we find that

$$(T^*TTT^*)T = \begin{bmatrix} 0 & 0 & 2 \\ 4 & 4 & 0 \\ 2 & -2 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 & 2 \\ 2 & 4 & 0 \\ 2 & -4 & 0 \end{bmatrix} = T(TT^*T^*T)$$

which prove that such an operator is not skew binormal one.

It is not the only one operator (matrix) with this property. Every matrix, real or complex, let denote by

$$T = \begin{bmatrix} 0 & 0 & a \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

is binormal, but not skew binormal. Indeed, calculating, with

$$T^* = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ \bar{a} & 0 & 0 \end{bmatrix}$$

we find that

$$T^*TTT^* = \begin{bmatrix} 2|a|^2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2|a|^2 \end{bmatrix} = TT^*T^*T$$

which proves binormality. On the other hand we have

$$(T^*TTT^*)T = \begin{bmatrix} 2|a|^2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2|a|^2 \end{bmatrix} \begin{bmatrix} 0 & 0 & a \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2a|a|^2 \\ 4 & 4 & 0 \\ 2|a|^2 & -2|a|^2 & 0 \end{bmatrix}$$

and

$$T(TT^*T^*T) = \begin{bmatrix} 0 & 0 & a \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2|a|^2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2|a|^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2a|a|^2 \\ 2|a|^2 & 4 & 0 \\ 2|a|^2 & -4 & 0 \end{bmatrix}$$

It is clear that, in general, $(T^*TTT^*)T \neq T(TT^*T^*T)$ which proves that such a matrix is not skew binormal one.

Example 2.2. The first part of this examples is in [4]. An operator can be skew binormal (skew $n-$ binormal), but need not to be binormal ($n-$ binormal). The matrix

$$T = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

is not 2-binormal because

$$T^*T^2T^2T^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = T^2T^*T^*T^2,$$

but is it a skew 2-binormal operator as shown this calculating

$$(T^*T^2T^2T^*)T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$T(T^2T^*T^*T^2) = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

That is $(T^*T^2T^2T^*)T = T(T^2T^*T^*T^2)$.

Taking a family of matrix (real or complex) represented by $T=\begin{bmatrix}0 & 1 & a\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}$, we can easily see that T is not 2-binormal operator, because

$$T^*T^2T^2T^* = \begin{bmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ \bar{a} & 1 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ \bar{a} & 1 & 0\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & \bar{a}\end{bmatrix}\begin{bmatrix}\bar{a} & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

and

$$T^2T^*T^*T^2 = \begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ \bar{a} & 1 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ \bar{a} & 1 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} = \begin{bmatrix}\bar{a} & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & \bar{a}\end{bmatrix} = \begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}.$$

On the other hand we find that

$$(T^*T^2T^2T^*)T = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 1 & a\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \text{ and}$$

$$T(T^2T^*T^*T^2) = \begin{bmatrix}0 & 1 & a\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$ no matter what the value of a is, which means skew 2-binormality.

We resume that, the condition at Theorem 2.2. [2] is weak to guarantee the skew binormality, and must be improved.

Our conclusions can be formulated in following propositions.

Proposition 2.3. *The class of binormal operators and skew binormal operators are independent.*

Proposition 2.4. *The class of $n-$binormal operators, [nBN], and skew $n-$binormal operators, [$snBN$], are independent.*

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