Journal of Natural Sciences and Mathematics of UT, Vol. 5, No. 9-10, 2020

UDC: 512.533.5:511.333
Professional paper

COMPLETELY PRIME IDEALS OF SEMIGROUPS

Merita Bajrami[1*], Rushadije Ramani-Halili[1]

[1]*Faculty of Natural Sciences and Mathematics, University of Tetova, Ilinden n.n., 1200 Tetovo, Republic of North Macedonia*

**Corresponding author e-mail: merita.azemi@unite.edu.mk , rushadije.ramani@unite.edu.mk*

Abstract

In this paper we discuss a very useful tool in the study of semigroup theory, called completely prime ideals. We will study characterization of semilattice congruences by means of completely prime ideals. We will follow the procedure of finding all subdirectly irreducible semilattices, then construct all congruences induced by homomorphism onto these, and finally take arbitrary intersections of congruences constructed. The usual algebraic books and papers studied semilattice congruence of semigroups by means of strongly prime ideals. We give some necessary and sufficient conditions about completely semiprime ideals, semilattice congruence and filters, which provide clearer informations about semigroups structure and new algebraic approach of studying them. Also we study the relationship between π-simple semigroup and completely prime ideals. So the purpose of this paper is two-fold: to characterize an arbitrary nonempty intersection of completely prime ideals and characterize subsets which are congruence classes of some semilattice congruence.

Keywords: Ideal, Prime, Semilattice, Congruence, Semigroup

1. Introduction

Completely prime ideals are studying more than any other part of semigroups. The purpose of this paper is to investigate the structural charasterictics of semigroups, such as semilattice congruences by means of completely prime ideals and filters.

The paper is organized as follows:

Firstly we give some definitions and lemmas we will use regarding the main result of a paper, according to [1], [2], [5], [6], [8] and [10]. In the second section, according to [4], [7] and [9] was shown the relationship between $\pi-$ simple semigroup and completely prime ideals. In the third section we give some necessary and sufficient conditions about completely semiprime semigroups, semilattice congruence and filters, which provide clearer information about semigroups structure and new algebraic approach of studying completely prime ideals.

Definition 1.1 [1, p.1] A semigroup is groupoid (S,*) such that the operation (*) is associative.

Definition 1.2 [6, p. 23] Let S be a semigroup. A nonempty subset I of S is a left(respectively right) ideal of S if for any $s \in S, a \in I$, we have $sa \in I$(respectively $as \in I$).

Further, I is a two-sided ideal(or simply an ideal) of S if it is both a left and a right ideal of S. A left (right, two-sided) ideal I of S is proper if $I \neq S$.

Definiton 1.3 [10, p.72] An ideal I fo a semigroup S is completely prime if for any $a, b \in S,\ ab \in I$ implies that either $a \in I$ or $b \in I$.

Definition 1.4 [5, p.28] A subsemigroup F of a semigroup S is a filter of S if for all $x, y \in S, xy \in F$ implies $x, y \in F$.

Lemma 1.5 [2, p.37] Let A be a non-empty subset of a semigroup S different from S. Then:

(i) A is a completely prime subset of S if and only if $S - A$ is a subsemigroup of S.

(ii) A is a completely prime left(right) ideal of S if and only if $S - A$ is a left(right) filter of S.

(iii) A is a completely prime ideal of S if and only if $S - A$ is a filter of S.

Lemma 1.6 [8, p.629] Let A be an ideal of a semigroup S. Then A is a prime ideal if and only if for ideals M, N of S, from $MN \subseteq A$ it follows that $M \subseteq A$ or $N \subseteq A$.

Proof:

Let A be a prime ideal of S, and let M and N be the ideals of S such that $MN \subseteq A$. Assume that there exist $x \in M - A$ and $y \in N - A$. Then $xSy \subseteq MSN \subseteq MN \subseteq A$, so $x \in A$ or $y \in A$, because A is a prime ideal. So it is a contradiction. Hence , $M - A = \varnothing$ or $N - A = \varnothing$, i.e $M \subseteq A$ or $N \subseteq A$

Conversely, for ideals M and N of S, from $MN \subseteq A$, let it follow that $M \subseteq A$ or $N \subseteq A$. Assume $x, y \in S$ suche that $xSy \subseteq A$. Then $J(x) \subseteq A$ or $J(\mathrm{y}) \subseteq A$, i.e. $x \in A$ or $y \in A$. Therefore A is a prime ideal of S .

2. π – simple semigroup and completely prime ideals.

Notation 2.1 For any element x of a semigroup S, let $N(x)$ denote the least filter of S containing x, and let $N_x = \{y \in S |\ N(x) = N(y)\}$.

Theorem 2.2 [7] If I is an ideal of an $\pi -$ class of a semigroup, then I has no proper completely prime ideals.

Proof:

Let S be any semigroup, z be an element of S an I be an ideal of N_z. It suffices to show that I, itself, is the only filter of I. Hence let F be a filter of I, a be an element of F, and let $T = \{x \in S | a^2x \in F\}$.

We show next that T is a filter of S.

Let $x, y \in T$; then $a^2y \in F$, which together with the inclusions $F \subseteq I \subseteq N_z$, implies $N_{ya} = N_{a^2y} = N_z$. Hence $ya \in N_z$ and thus $ya^2 \in I$. Further, $a^2(ya^2) = (a^2y)a^2 \in F$ so that $ya^2 \in F$.

Similarly, $a^2x \in F$ implies $ax \in N_z$, which in turn yields $axy \in N_z$ since $N_{axy} = N_{ax}N_{ay} = N_z$. Consequently $a^2xy \in I$; on the other hand, $a^2x, ya^2 \in F$ implies $(a^2x)(ya^2) \in F$. But then also $(a^2xy)a^2 \in F$ so that $a^2xy \in F$. Therefore $xy \in T$.

Conversely, let $xy \in T$. Then $a^2xy \in F$, and thus also $(a^2x)(ya^2) = (a^2xy)a^2 \in F$. Moreover, $a^2xy \in F$ easily implies $ax, ya \in N_z$ and hence $a^2x, ya^2 \in I$. But since $(a^2x)(ya^2) \in F$, it follows that $a^2x, ya^2 \in F$. As before, we infer that $a^2y \in F$ since $(a^2y)a^2 = a^2(ya^2) \in F$ and $a^2y \in I$. Thus $x, y \in T$.

Hence T is a filter of S. It is easy to see that $T \cap I = F$. But then $a \in N_z \cap T$ implies that $N_z \subseteq T$ and therefore $T \cap I = I$, i.e., $F = I$.

Definition 2.3 [9, p.32] A semigroup without proper completely prime ideals is π-simple.

Corollary 2.4 [9, p.32] Every completely prime ideal of a semigroup S is a union of π-classes.

Proof:

If I is a completely prime ideal of a semigroup S, then $J = \{N_x \in Y_S | x \in I\}$ is a completely prime ideal of Y_S, then $I = \{x \in S | N_x \in J\}$ is a completely prime of S. This establishes a one-to-one, order-preserving(relative to inclusion) correspondence between the partially ordered set of all completely prime ideals of S and the partially ordered set of all completely prime ideals of Y_S.

Definition 2.5 [4, p.212] Let S be a semigroup. An ideal $I$of S is prime if for any $a, b \in S, aSb \subseteq I$ implies that either $a \in I$ or $b \in I$; I is semiprime if and only if for any $a \in S, aSa \subseteq I$ implies $a \in I$.

A nonempty subset A of S is completely semiprime if for any $x \in S$, $x^2 \in A$ implies $x \in A$; A is a m-system if for any $a, b \in A$ there exist $x \in S$ such that $axb \in A$.

Corollary 2.6 [9, p.36] Every semiprime ideal is the intersections of minimal prime ideals containing it.

Proof:

Let I be a semiprime ideal of a semigroup S, and let $d \in S \backslash I$. Letting

$$M = \{d^n | n = 1,2, \dots\}$$

we obtain an $m-$system disjoint from I; by virtue of [13, p.36] there exist a minimal prime ideal P containing I and not containing d.

3. Main results

Theorem 3.1 The following conditions on an ideal I of a semigroup S are equivalent.

i) I is the intersection of completely prime ideals.
ii) I is the intersection of minimal completely prime ideals containing it.

iii) I is the union of $\pi -$ classes.
iv) I is completely semiprime.

Proof:

Let $I = \bigcap_{\alpha \in A} I_\alpha$ $I = \bigcap_{\alpha \in A} I_\alpha$ where each I_α is a completely prime ideal. Fix $\alpha \in A$ and let $\mathbf{T}$ be the partially ordered set of all completely prime ideals J of S for which $\subseteq J \subseteq I_\alpha$. Then $\mathbf{T} \neq \emptyset$ since $I_\alpha \in \mathbf{T}$. Let $\mathbf{e}$ be a chain in $\mathbf{T}$ and let $A = \bigcap_{C \in e} C$. Then $I \subseteq A \subseteq I_\alpha$ and A is an deal of S. Since the partially orderd set $S \setminus A = \bigcup_{C \in e} (S \setminus C)$ is a subsemigroup of S or is empty, which shows that A is a completely prime ideal.

Hence $A \in \mathbf{T}$ and the minimal principle assures the existence of a minimal element, say J_α, in $\mathbf{T}$. But then J_α is also minimal relative to the property of being a completely prime ideal containing I. In addition

$$I \subseteq \bigcap_{\alpha \in A} J_\alpha \subseteq \bigcap_{\alpha \in A} I_\alpha = I$$

So $\bigcap_{\alpha \in A} J_\alpha = I$ and we have that $(i) \Rightarrow (ii)$.

From 2.4 follows that $(ii) \Rightarrow (iii)$.

If $x^2 \in I$, then $N_x = N_{x^2} \subseteq I$ so that $x \in I$ and $(iii) \Rightarrow (iv)$.

By 2.6, for every $d \notin I$, there exist a minimal prime ideal J containing I and not containing d.

According to 2.3, J is completely prime. Thus I is the intersection of all completely prime ideals containing it.

So $(iv) \Rightarrow (i)$.

Definition 3.2 A nonempty subset C of a semigroup S is an $\pi -$ subset if C is completely semiprime and satisfies the condition: for any $x, y \in S$ and $z \in S^1$, $x, yz \in C$ implies $xy, zx \in C$.

Lemma 3.3 [3, p.189] Let C be an π-subset of a semigroup S. Then for any $a, b \in S$ and $x \in S^1$, $xab \in C$ implies $xba \in C$.

Proof:

Suppose first $c, d \in S, cd \in C$. Then $cd, cd \in C$ implies $d(cd), cd \in C$ whence $(dc)^2 = d(cd)c \in C$. Thus $cd \in C, dc \in C$.

Next let $x, a, b \in S$ with $xab \in C$. Then $(ab)x, b(xa) \in C$ so that $a(bxb) = (ab)xb \in C$. Consequently $(xbax)b, a(bx) \in C$ so that $(xba)^2 = (xbax)ba \in C$ and hence $xba \in C$.

Theorem 3.4 The following conditions on a nonempty subset C of a semigroup S are equivalent.

i) C is an $\pi -$ subset.
ii) C is a class of a semilattice congruence.
iii) C is the intersection of a completely semiprime ideal and a filter.

Proof:

$i)$ Implies $ii)$.

Define a r elation σ on S by:

$a \, \sigma \, b$ if for every $x \in S^1, xa \in C$ if and only if $xb \in C$. It is clear that σ is an equivalent relation and a left congruence. Let $a \, \sigma \, b$ and $c \in S$. Let $x \in S^1$. If $x(ac) \in C$ implies $x(bc) \in C$ and conversely by symmetry.

Hence σ is a right congruence and thus a congruence. Further, S/σ is commutative.

Next let $x \in S$. If $xa \in C$, then $xa, ax \in C$ and hence $xa^2 \in C$ since C is an $\pi -$ subset.

Conversely, if $xa^2 \in C$ then xa^2 implies $xa^2x \in C$ and thus $(xa)^2 \in C$. But then $xa \in C$ since C is completely semiprime. Further, $a \in C$ if and only if $a^2 \in C$ since C is an $\pi -$ subset. Consequently $a \, \sigma \, a^2$ which implies that S/σ is idempotent. Hence according to 3.1 σ is a semilattice congruence.

If $a \in C$ and $a\sigma b$, then $b \in C$, which implies that C is a union of $\sigma -$ classes.

Let $a, b \in C$. If $xa \in C$, then $b, xa \in C$ so that $bx \in C$ and thus also $xb \in C$; by symmetry, $xb \in C$ implies $xa \in C$. Hence $a \, \sigma \, b$, and we deduce that C is a $\sigma -$ class.

$ii)$ Implies $iii)$

Let C be a class of a semilattice congruence τ, let $S = \bigcup_{\alpha \in Y} S_\alpha$ be the corresponding decomposition of S, and let $C = S_\gamma$. Letting

$$A = \bigcup_{\alpha \le \gamma} S_\alpha \text{ and } B = \bigcup_{\alpha \ge \gamma} S_\alpha ,$$

it is easy to verify that A is a completely prime ideal and B is a filter such that $A \cap B = S_\gamma = C$.

$iii)$ Implies $i)$

Let $C = A \cap B$ where A is a completely semiprime ideal and B is a filter. Since both A and B are completely semiprime, so is C. If $y, z \in S$ and $x, yz \in C$, then $xy, zx \in A$ and $x, y, z \in B$ so that $xy, zx \in B$ and thus $xy, zx \in C$. If $x, y \in C$, then clearly $xy, x \in C$. Hence C is an $\pi -$ subset.

References

[1]. Clifford, A. H., & Preston, G. B. (1967). *The algebraic theory of semigroups, Volume II* (Vol. 2). American Mathematical Soc.

[2]. Bogdanovic, S., Ciric, M., &Popovic, Z. L. (2011). Semilattice decompositions of semigroups. *University of Niš, Faculty of Economics*.

[3]. Mitrović, M., Bogdanović, S., & ĆIRIĆ, M. (2000). Locally uniformly π-regular semigroups. In *Semigroups* (pp. 106-113).

[4]. Rao, V. S. (1982). A characterization of semiprime ideals in near-rings. *Journal of the Australian Mathematical Society*, *32*(2), 212-214.

[5]. Zhu, Q., & Popović, Ž. (2016). Decompositions of Ordered Semigroups into a Chain of k-Archimedean Ordered Semigroups. *Filomat*, *30*(10), 2771-2780.

[6]. Petrich, M. (1984). *Inverse semigroups* (No. 1). Wiley-Interscience.

[7]. Bell, A. D., Stalder, S. S., & Teply, M. L. (1996). Prime ideals and radicals in semigroup-graded rings. *Proceedings of the Edinburgh Mathematical Society*, *39*(1), 1-25.

[8]. Park, Y. S., & Kim, J. P. (1992). Prime and semiprime ideals in semigroups. *Kyungpook Mathematical Journal*, *32*(3_spc), 629-633.

[9]. Petrich, M. (1973). Introduction to semigroups. *Chares E.*

[10]. Schwarz, Š. (1969). Prime ideals and maximal ideals in semigroups. *Czechoslovak Mathematical Journal*, *19*(1), 72-79.