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Professional paper

STOCHASTIC DIFFERENTIAL EQUATIONS SIMULATION OF EULER-MARUYAMA APPROXIMATION METHOD

Bilall I. Shaini[1] , Dashmir Ibishi[2], Shpëtim Rexhepi[3]

[1,2]*Department of Mathematics, University of Tetova, North Macedonia*
[3]*Department of Mathematics, University Mother Tereza, Skopje, North Macedonia*
[*]*Corresponding author e-mail: bilall.shaini@unite.edu.mk*

Abstract

This paper is an introduction and survey of numerical solution methods for stochastic differential equations. In mathematics and computational sciences, the Euler method is a first-order numerical procedure for solving ordinary differential equations (ODEs) with given initial value, in SDEs Euler approximations give one of the best results after working with an approximation method.
Also, to the Stochastic Differential Equations (SDEs) one of the simplest time discrete approximation of an Ito process is the Euler approximation, or called the Euler-Maruyama approximation. We shall consider an Ito process $X = \{X_t, t_0 \le t \le T\}$ satisfying the scalar of a SDE. And to illustrate various aspects of the simulation of a time discrete approximation of an Ito process we shall examine a simple example by using Excel Simulation and MathLab Simulation.

Keywords*:* SDEs, Euler-Maruyama approximation, Ito process, Excel Simulation, MathLab Simulation

1. Introduction

What are Ito Stochastics?

The stochastic calculus of a Ito originated with his investigation of conditions under which the local properties of a Markov process could be used to characterize this process. By local properties here we mean quantities such as the drift and the diffusion coefficient of a diffusion process. These had been used some time earlier by Kolmogorov to drive the partial differential equations, which now bear his name.
In contrast, Ito's approach focused on the functional formulation of stochastic differential equations, which until had been inadequate.
An ordinary differential equation

$$\dot{x} = \frac{dx}{dt} = a(t, x)$$

May be thought of as a degenerate form of a stochastic differential equation, as undefined and as a randomnees. We can write (1) as a integral equation

$$x(t) = x_0 + \int_{t_0}^{t} a(s, x(s))ds$$

Where $x(t) = x(t; x_0; t_0)$ is a solution satisfying the initial condition $x = x_0$.

2. Brownian motion

2.1. Definition of Brownian motion. Brownian motion is closely linked to the normal distribution. Recall that a random variable *X* is normally distributed with mean μ and variance σ^2 if

$$P\{X > x\} = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{x}^{\infty} e^{-\frac{(u-\mu)^2}{2\sigma^2}} \quad \text{for all } x.$$

Definition 2.2

A real-valued stochastic process $\{B(t) : t \geq 0\}$ is called a (linear)

Brownian motion with start in $x \in R$ if the following holds:

- $B(0) = x$
- the process has independent increments, i.e. for all times $0 \leq t_1 \leq t_2 \leq ... \leq t_n$ the increments $B(t_n) - B(t_{n-1}), B(t_{n-1}) - B(t_{n-2}), ..., B(t_2) - B(t_1)$ are independent random variables,
- for all $t \geq 0$ and $h > 0$, the increments $B(t+h) - B(t)$ are normally distributed with expectation zero and variance h
- almost surely, the function $t \mapsto B(t)$ is continuous.

We say that $\{B(t) : t \geq 0\}$ is a standard Brownian motion if $x = 0$.

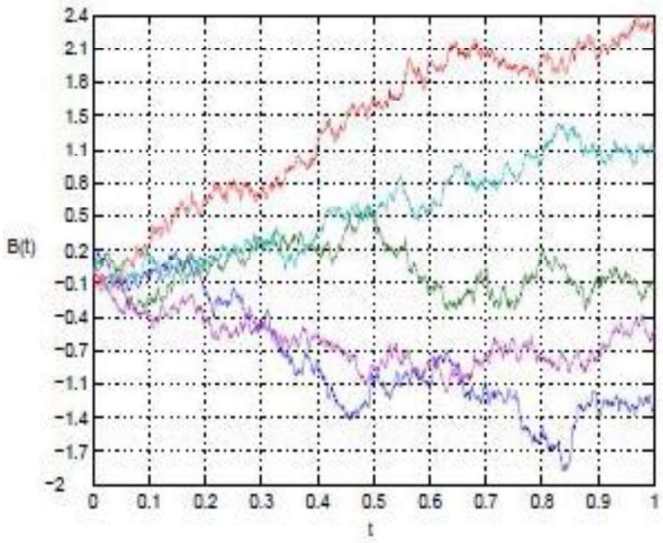


Figure 1. Graphs of five sampled Brownian motions

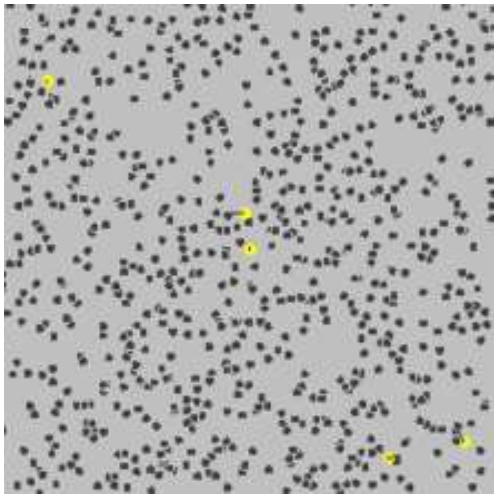


Figure 2. Nature Brownian Motions

3. The Euler Approximation

One of the simplest time discrete approximations of an Ito process is the Euler approximation or the Euler-Marutama approximation as it sometimes called. We shall consider an Ito process $X = \{X_t, t_0 \le t \le T\}$ satisfying the scalar stochastic differential equation

$$dX_t = a(X_t)dt + b(X_t)dW_t$$

(3.1)

On to $t_0 \le t \le T$ with the initial value

$$X_{t_0} = X_0$$

(3.2)

For a given discretization $t_0 = \tau_0 < \tau_1 < ... < \tau_n < ... < \tau_N = T$ of the time interval $[t_0, T]$, an Euler approximation is continuous time stochastic process $Y = \{Y(t), t_0 \le t \le T\}$ satisfying the iterative scheme

$$Y_{n+1} = Y_n + \alpha(\tau_n, Y_n)(\tau_{n+1}, \tau_n) + b(\tau_n, Y_n)\left(W_{\tau_{n+1}} - W_{\tau_n}\right) \text{ for } n = 0,1,2,\ldots, N\text{-}1$$

(3.3)

With initial value $Y_0 = X_0$

Where we have written $Y_n = Y(\tau_n)$, and for the value of the approximation at the discretization time τ_n. We shall also write

$$\Delta_n = \tau_{n+1} - \tau_n$$

(3.4)

For the $n-th$ time increment and call like:

$$\delta = \max \Delta_n \text{ the maximum time step}$$

(3.5)

When the diffusion coefficient is identically zero, that is when $b \square 0$, the stochastic iterative schema (3.3) reduces to the deterministic Euler schema (3.1) for the ordinary equation

$\frac{dx}{dt} = a(t, x)$, and the sequence $\{Y_n, n = 0,1,2,..., N\}$ of the Euler approximation (1.3) at the instants of the time discretization $(\tau)_\delta = \{n = 0,1,..., N\}$ can be computed in a similar way to those of the deterministic case.

Time discrete simulation

To illustrate various aspects of the simulation of a time approximation of an Ito process we shall examine a simple example in some detail. We shall consider an Ito process $X = \{X_t, t_0 \le t \le T\}$ satisfying the scalar stochastic differential equation

$$dX_t = a(X_t)dt + b(X_t)dW_t$$

for $t \in [0, T]$ with the initial value $X_0 \in \Re$.

Application

Now we will see how dose a simulation work in excel:

We will generate equidistant Euler approximations on the time interval [0,1] with equal step size $\Delta = 2^{-2}$ for the Ito process **X** satisfying (3.1) with $X_0 = 1.0$ and $a = 1.5$ $b = 1.0$.

Euler method approximation to:

$dX_t = a X_t\, dt + b X_t\, dW_t$

(end time) T=	1
(time step) Dt=	0,002
initial value) X0=	1
drift parameter) a=	01.jaH
sion parameter) b=	1

X_T = 3,30138

t	euler X_t
0	1
0,002	1,083341433
0,004	1,085911511
0,006	1,192418756
0,008	1,187701077
0,01	1,230960873
0,012	1,205927381
0,014	1,132166825
0,016	1,193887269
0,018	1,206974955
0,02	1,268120564
0,022	1,320383906
0,024	1,366265542
0,026	1,456546296
0,028	1,352530459
0,03	1,411761734
0,032	1,367578243
0,034	1,333754031
0,036	1,23982856
0,038	1,314368249
0,04	1,449603087
0,042	1,411276635
0,044	1,420440301
0,046	1,397637076
0,048	1,35073028
0,05	1,302887406
0,052	1,239722845
0,054	1,256974518
0,056	1,271741439
0,058	1,334985336
0,06	1,256981225
0,062	1,212271699
0,064	1,233292415
0,066	1,230311047
0,068	1,201667103
0,07	1,256121856
0,072	1,230317381
0,074	1,339918954

3.1 Euler simulation with 200 points, in Excel

4. Strong and Weak Convergence of the EM Method.

In the example above with em.m the EM solution matches the true solution more closely as Δt is decreased—convergence seems to take place. Keeping in mind that $X(\tau n)$ and Xn are random variables, in order to make the notion of convergence precise we must decide how to measure their difference. Using $E\,|\,Xn - X(\tau n)\,|$, where E denotes the expected value, leads to the concept of strong convergence. A method is said to have *strong order of convergence* equal to γ if there exists a constant C such that:

$$\mathrm{E}|Xn - X(\tau)| \le C\Delta t^{\gamma} \tag{4.1}$$

for any fixed $\tau = n\Delta t \in [0,\ T]$ and Δt sufficiently small. If f and g satisfy appropriate conditions, it can be shown that EM has strong order of convergence $\gamma = \frac{1}{2}$

Note that this marks a departure from the deterministic setting—if $g \equiv 0$ and X_0 is constant, then the expected value can be deleted from the left-hand side of (4.1) and the inequality is true with $\gamma = 1$.

In our numerical tests, we will focus on the error at the endpoint $t = T$, so we let *e*strong

$$e_{\Delta t}^{\ strong} = E\left|X_L - X(T)\right|, \quad \text{where} \quad L\Delta t = T \tag{4.2}$$

denote the EM endpoint error in this strong sense. If the bound (5.1) holds with $\gamma = \frac{1}{2}$ at any fixed point in $[0,\ T]$, then it certainly holds at the endpoint, so we have

$$e_{\Delta t}^{\ strong} \le C\Delta t^{1/2} \tag{4.3}$$

for sufficiently small Δt.

The M-file emstrong.m in Listing 6 looks at the strong convergence of EM for the SDE using the same λ, μ, and X_0 as in em.m. We compute 1000 different discretized Brownian paths over $[0,\ 1]$ with $\delta t = 2^{-9}$. For each path, EM is applied.

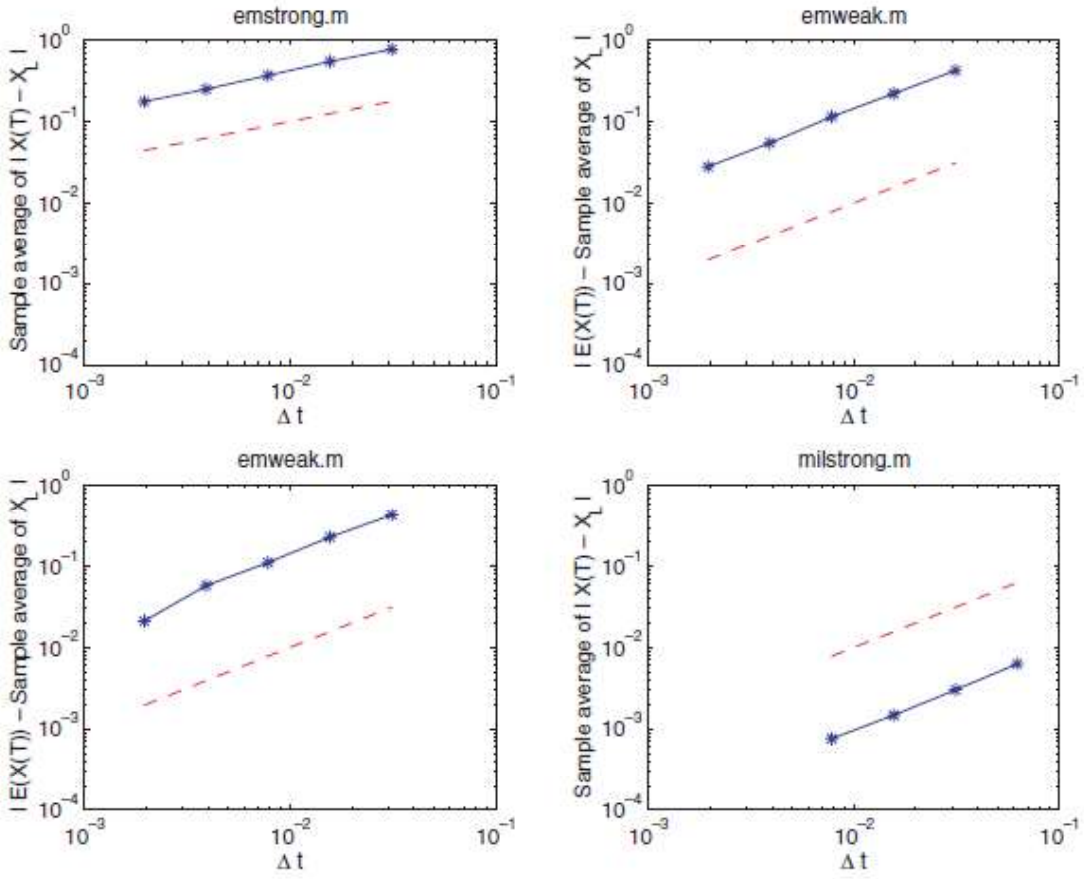
```
%EMSTRONG  Test strong convergence of Euler-Maruyama
%
% Solves     dX = lambda*X dt + mu*X dW,    X(0) = Xzero,
%          where lambda = 2, mu = 1 and Xzero = 1.
%
% Discretized Brownian path over [0,1] has dt = 2^(-9).
% E-M uses 5 different timesteps: 16dt, 8dt, 4dt, 2dt, dt.
% Examine strong convergence at T=1:  E | X_L - X(T) |.

randn('state',100)
lambda = 2; mu = 1; Xzero = 1;     % problem parameters
T = 1; N = 2^9; dt = T/N;          %
M = 1000;                          % number of paths sampled

Xerr = zeros(M,5);                 % preallocate array
for s = 1:M,                       % sample over discrete Brownian paths
    dW = sqrt(dt)*randn(1,N);      % Brownian increments
    W = cumsum(dW);                % discrete Brownian path
    Xtrue = Xzero*exp((lambda-0.5*mu^2)+mu*W(end));
    for p = 1:5
        R = 2^(p-1); Dt = R*dt; L = N/R;      % L Euler steps of size Dt = R*dt
        Xtemp = Xzero;
        for j = 1:L
            Winc = sum(dW(R*(j-1)+1:R*j));
            Xtemp = Xtemp + Dt*lambda*Xtemp + mu*Xtemp*Winc;
        end
        Xerr(s,p) = abs(Xtemp - Xtrue);       % store the error at t = 1
    end
end

Dtvals = dt*(2.^([0:4]));
subplot(221)                                   % top LH picture
loglog(Dtvals,mean(Xerr),'b*-'), hold on
loglog(Dtvals,(Dtvals.^(.5)),'r--'), hold off % reference slope of 1/2
axis([1e-3 1e-1 1e-4 1])
xlabel('\Delta t'), ylabel('Sample average of | X(T) - X_L |')
title('emstrong.m','FontSize',10)

%%%% Least squares fit of error = C * Dt^q %%%%
A = [ones(5,1), log(Dtvals)']; rhs = log(mean(Xerr)');
sol = A\rhs; q = sol(2)
resid = norm(A*sol - rhs)
```



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