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Review article

PARAMEDIAL ARCHIMEDEAN SEMIGROUPS

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Abstract

In this paper we investigate paramedial Archimedean semigroups with idempotent. Firstly we give some general notations, definitions and auxiliary facts related to semigroups. A semigroup S is called Archimedean if and only if for all $a, b \in S$, there exist $u, v \in S$ and positive integers n, m such that $a^n = bu$ and $b^m = av$. A paramedial semigroup is a semigroup S satisfying the paramedial law $abcd = dbca$ for all $a, b, c, d \in S$.

***Keywords*:** Semigroup, paramedial, Archimedean, rectangular group

1.Introduction

Petar V. Protiç [2] and N. Sawatraska; Ch. Namnak [3] introduced the concept of paramedial semigroups. They investigated that some general properties of paramedial and medial semigroups and semilattice decomposition of paramedial and medial semigroups.
We present a number of definitions and notations most of which will be indispensable for our research.

Definition 1. A semigroup is a set S together with a binary operation " • " that satisfies the associative property: For all $a, b, c \in S$ the equation $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ holds.

Definition 2. A direct product of a rectangular band and a group is called a rectangular group.

Definition 3. A semigroup S is called Archimedean if and only if for all $a, b \in S$ there exist $u, v \in S$ and positive integers n, m such that $a^n = bu$ and $b^m = av$.

Definition 4. A paramedial semigroup is a semigroup S satisfying the paramedial law $abcd = dbca$ for all $a, b, c, d \in S$.

Lema1.[1]: Every paramedial semigroup is medial.

Definition 5. Let S be an Archimedean semigroup without idempotent.

We define a congruence $\sigma_b$on S for fixed $b \in S$ as follows.We define $x\sigma_b y$ if and only if there are positive integers n and m such that $b^n x = b^m y$.

Definition 6. An idempotent in a semigroup S is an element e such that $e^2 = e$. The set of idempotents of S is denoted by $E(S)$. Two special idempotents are the identity element, if it exists, and the zero element, if it exists.

Theorem 1[1]. A semigroup is a rectangular group if and only if it is a completely simple semigroup in which the idempotents form a subsemigroups.

Theorem 2.[1] A variety of semigroup is closed with respect to retract extension.

Theorem 3.[1] A semigroup S is Archimedean and contains at least one idempotent element if and only if it is an ideal extension of simple semigroup containing an idempotent by a nil semigroup.

Theorem 4.[2]: A semigroup is paramedial and 0-simple if and only if it is a commutative group with a zero adjoined.

Theorem 5.[3] A semigroup S is a medial Archimedean semigroup containing at least one idempotent element if and only if it is a retract extension of a rectangular abelian group by a medial nil-semigroup.

2.Paramedial Archimedean semigroup

In this section we investigate paramedial Archimedean semigroups containing idempotents.

Proposition 1. For a paramedial semigroup P , $a\sigma_P b (a, b \in P)$ if and only if $a^n \in PbP$ and $b^n \in PaP$ for some $n \in \mathbb{N}$.

Proof: Define a relation ρon P by $a\rho b$if $a^n \in PbP$ and $b^n \in PaP$ for some $n \in \mathbb{N}$.

Refleksivity and symmetry of ρ are obvious.

No we proof transitivity and compatibility:

Let $a\rho b$, $b\rho c$ and $d \in P$. Then $a^n \in PbP$,$b^n \in PaP$,$b^m \in PcP$ and $c^m \in PbP$

for some $m, n \in \mathbb{N}$.

Thus $a^{nm} \in (PbP)^m = P^m b^m P^m \subseteq P^m PcPP^m \subseteq PcP$.

Similarly, $c^{mn} \in PaP$. Thus $a\rho c$.

The proof that $\rho$is compatible involves four statements like the following:

$(ad)^{n+1} = a^{n+1} d^{n+1} \in PbPd^{n+1} \subseteq PbdPd^n \subseteq PbdP$.

It now follows easily that ρ is a semilattice congruence, for $x^4 \in Px^2P, (x^2)^2 \in PxP, (xy)^2 \in PyxP$ and $(yx)^2 \in PxyP$ for every $x, y \in P$.

Hence, $\rho_P \subseteq \rho$.

Conversely, if $a\rho b$ then $a^n = ubv$ and $b^n = waz$ for some $n \in \mathbb{N}$ and $u, v, w, z \in P$. Thus

$a\sigma_P a^n \sigma_P ubv \sigma_P ub^2 v \sigma_P ubvb \sigma_P a^n b^n \sigma_P awaz \sigma_P waz \sigma_P b^n \sigma_P b$. Therefore $\rho \subseteq \sigma_P$. Hence $\rho = \sigma_P$.

It rimeans to show that a congruence class L of σ_P is an Archimedean semigroup. By what was mentioned in the introduction, $a\sigma_L b$ for every $a, b \in L$. But L is paramedial semigroup and we therefore can apply the explicit formulation of σ_L above. Thus L is Archimedean.

For paramedial semigroup P, we will call the congruence classes of P modulo σ_P the Archimedean components of P and will say that P is a semilattice of Archimedean semigroups and write

$$P = \cup \{P_\alpha | \alpha \in Y\}$$

where $Y = P/\sigma_P$ and $P_\alpha P_\beta \subseteq P_{\alpha\beta}$.

Proposition 2. Let P be a paramedial semigroup. P is an Archimedean semigroup possessing an idempotent if and only if P contains an ideal I which is both a rectangular group and a root of P.

Proof: Let P be Archimedean with an indempotent. By proposition1, P contains a simple ideal I which is a root of P.

Let x, y be idempotents of I such that $xy = yx = x$. Since I is simple, $y = exf$ for some $e, f \in I$.

Thus $y = y^2 = exfy = ex^2fy = exfxy = yxy = xy = x$

So every idempotent of I is primitive, and hence I is a completely simple semigroup. But the idempotents of a paramedial semigroup form a subsemigroup.

So, I is a rectangular group

Since a rectangular group is simple with an idempotent, the converse is a trivial consecuence of proposition1.

Corollary1. Let P be a paramedial semigroup and P is an Archimedean semigroup and I ideal, which is a root of P.

If E is the set of idempotents of P, then E is a rectangular band and xPy is an abelian group for all $x, y \in E$ and $I \cong xPy \times E$ for all $x, y \in E$.

Proof: All idempotents of P are in I. But if $I \cong G \times \Gamma$,G -group and Γ - a rectangular band, then Γ is isomorphic to the set of all idempotents of I.

Hence E is isomorphic to the rectangular band Γ. Furthermore, since P is paramedial xPy is abelian.

However, $xIy \cong G$. Thus $I \cong xPy \times E$ for all $x, y \in E$.

Lema 2. A semigroup is paramedial and Archimedean containing at least one idempotent element if and anly if it is ideal exstension of a commutative group by an paramedial nil-semigroup.

Proof: Let S be a paramedial Archimedean semigroup containing at least one idempotent element.

As S is also a medial by lema 1.[3] it is an ideal extension of a rectangular abelian group J by a medial nil-semigroup Q.

(see theorem 5[3]). Since J is simple and paramedial then by theorem 4[2] it is a commutative group. It is evident that Q is also a paramedial semigroup.

Conversely, assume that the semigroup S is an ideal extension of a commutative group G by an paramedeal nil-semigroup Q.

Then by theorem 3[1], S is an Archimedean semigroup with an idempotent element. It is easy to see that $\Phi: s \mapsto es$ is a retract homomorphism of S onto G,where e denotes the identity of G.As the paramedial semigroup form a variety, S is paramedial (see theorem 2.[1]).

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