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*Professional paper*

# STABILITY ANALYSIS FOR ONE-DIMENSIONAL NONLINEAR DYNAMICAL SYSTEMS WITH MATHEMATICA

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**Abstract**

Stability, like one of the most important concepts in Discrete Dynamical Theory, can tell much about the behavior of the dynamical system. In this article a symbolic Mathematica package for analysis and control of chaos in discrete one dimensional dynamical nonlinear systems, is presented. There are constructed some computer codes to find stability types of the fixed points, covering the stability of the one-dimensional nonlinear dynamical systems. Applications are taken from chemical kinetics and population dynamics (logistic model). To get a better understanding of the dynamics involved, we analyze examples using the Cobweb diagram, Phase Portrait and Time Series solution coded for one dimensional nonlinear dynamical systems.

***Keywords*****:** dynamical systems, fixed points, CobWeb diagram, Phase portrait and Time series, logistic model

## 1. Introduction

Dynamical modeling is the study of change and changes take place everywhere in life. As a result dynamical systems have a wide range of application areas in applied science and engineering. With these systems, real life situations can be turned into the language of mathematics. If one can create a good model for a real life situation, we will be able to predict the future states of the system by simply iterations according to this model. We assume that a given real dynamical process has to be studied, first we identify the state variables of the system. For example, in the biological process involving the two competing species of whiteflies and black wasps, we are interested in knowing how each species affects the evolution of the other. We identify possible control parameters of the system. In this step we look for those parameters that affect the evolution of the state variables. The next step is to determine the mathematical relations that translate the laws governing the evolution of the process [2,3].
Mathematica is extremely popular with a wide range of researchers from all sorts of disciplines. It is a symbolic, numerical and graphical manipulation package. This paper is both an survey on theory and techniques of discrete dynamical systems by using of the software Mathematica [1,3].

## 2. Stability analysis of one-dimensional nonlinear dinamical systems

A population is defined as a group of individuals of the same species within a limited area. Mathematical models are used to predict the size or density (population size per unit area) of a population at any time in the future [3]. They are also used to check the biological assumptions that are made to produce the model. Let $x_t$ be the size (density) of a population at time $t$ , and $x_{t+1}$ be the size (density) of this population at the next time interval or generation [3]. Then $x_{t+1}$ is related to $x_t$ by a function $f$ which may be written in the form $x_{t+1} = f(x_t)$.

*Definition 2.1.* [2] Let $f : R \to R$ be a map. Take an initial point $x_0 \in R$ and iterate this point with respect to the map $f$ . The set of values found with iterations give us the orbit $O(x_0)$ of the point $x_0$ . That is $O(x_0) = \{x_0, f(x_0), f^2(x_0), f^3(x_0), \ldots\}$ where $f^n = \underbrace{f \circ f \circ \ldots \circ f}_{n-times}$.

If we let $x_n = f^n(x_0)$, then a discrete dynamical system is a relation of the form

$$x_{n+1} = f(x_n) \tag{2.1}$$

*Definition 2.2.* [2] A point $\bar{x}$ is said to be fixed point of (2.1) if $f(\bar{x}) = \bar{x}$.

*Definition 2.3.* [2] Let $f : I \to I$ be a map and $\bar{x}$ be a fixed point of $f$ , where $I$ is an interval of $R$ . Then $\bar{x}$ is said to be stable if for any $\varepsilon > 0$, there exist $\delta > 0$ such that for all $x_0 \in I$ with $|x_0 - \bar{x}| < \delta$ we have $|f(x_0) - f(\bar{x})| < \varepsilon$ for all $n \in N$.

*Theorem 2.1.* Let $f$ be a map on $R$ which is continuously differentiable at $\bar{x} \in R$. And let $\bar{x}$ be hyperbolic fixed point of the map $f$ that is $|f'(\bar{x})| \neq 1$. Than if $|f'(\bar{x})| < 1$ then $\bar{x}$ is asymptotically stable and if $|f'(\bar{x})| > 1$ then $\bar{x}$ is unstable [3].

*Theorem 2.2.* For nonhyperbolic fixed points [1]

- If $|f'(\bar{x})| = 1$ ( $f', f'', f'''$ are continuous functions)
  a) If $f''(\bar{x}) \neq 0$, then $\bar{x}$ is unstable (semistable)
  b) If $f''(\bar{x}) = 0 \wedge f'''(\bar{x}) > 0$, then $\bar{x}$ is unstable
  c) If $f''(\bar{x}) = 0 \wedge f'''(\bar{x}) < 0$, then $\bar{x}$ is asymptotically stable
- If $|f'(\bar{x})| = -1$ ( $f', f'', f'''$ are continuous functions)
  a) If $Sf(\bar{x}) < 0$, then $\bar{x}$ is asymptotically stable and $Sf(\bar{x}) = \frac{f'''(\bar{x})}{f'(\bar{x})} - \frac{3}{2}\left(\frac{f''(\bar{x})}{f'(\bar{x})}\right)^2$
  d) If $Sf(\bar{x}) > 0$, then $\bar{x}$ is unstable

## 3. Applications and Conclusions

*Example 3.1. (Logistic Model)*
Investigate the general logistic model [3] given by

$$\frac{dx}{dt} = \mu x(t)(1 - x(t-1)) \tag{3.1}$$

with initial history function $x(t) = 0.1$ on $[-1,0]$. The stability investigation will be done by using geometrical interpretation of time series and phase portrait. There are two critical points at $\bar{x} = 0$ and $\bar{x} = 1$. One can show that the trivial critical point at $\bar{x} = 0$ is unstable. Consider the critical point at $\bar{x} = 1$, where more interesting behavior is present. The characteristic equation [2] is given by $\mu e^{-\lambda} + \lambda = 0$. For the critical point $\bar{x} = 1$ to be stable, the complex roots of characteristic equation must lie on the left-half of the $\lambda$ plane

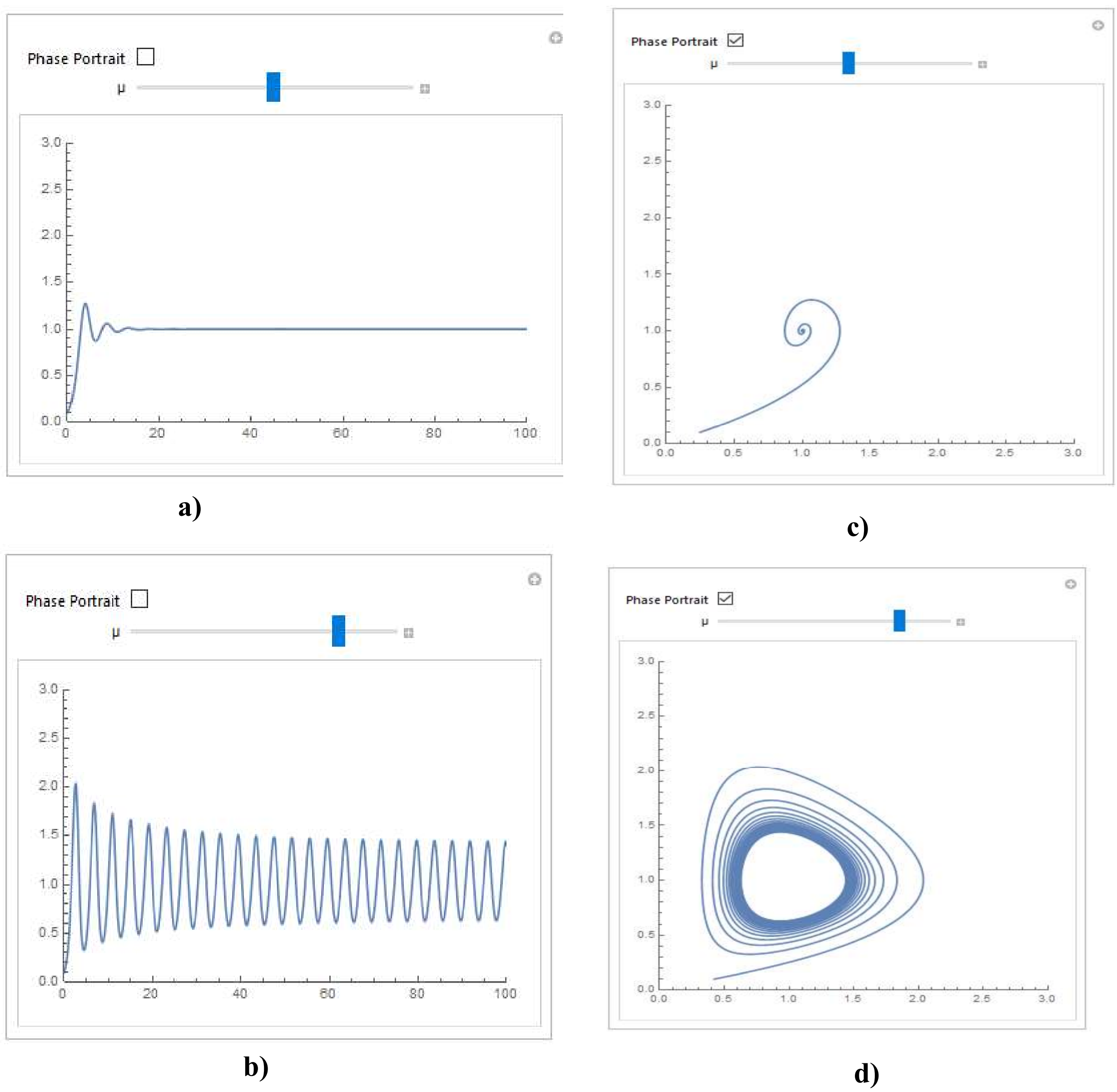


**Figure 1.** Time Series and Phase Portrait

Solutions to the equation (3.1) with initial history function on [−1, 0] are defined by $x(t) = 0.1$ for varying values of the parameter $\mu$ are: **a)** Time series showing that when $\mu = 1$, $x(t)$ approaches the stable critical point at $\bar{x} = 1$. **b)** Time series showing that when $\mu = 1.6$, $x(t)$ approaches a stable limit cycle. **c)** Phase portrait showing that when $\mu = 1$, $x(t)$ approaches the stable critical point at $\bar{x} = 1$. **d)** Phase portrait showing that when $\mu = 1.6$, $x(t)$ approaches a stable limit cycle. Note that the Manipulate command is used to produce an animation in the notebook for values of $\mu$ from 0 to 2.

*Example 3.2. (Simple Logistic Model)*

The differential equation $\frac{dx}{dt} = ax$ can be considered simple model of population growth when $a > 0$ . The quantity $x(t)$ measures the population of some species at time $t$. The assumption that leads to the differential equation is that the rate of growth of the population is directly proportional to the size of population [2]. One differential equation that satisfies the assumptions

1. If the population is small, the growth rate remains directly proportional to the size of the population
2. If the population growths too large, however, the growth rate becomes negative.

is the logistic population model [3], with the form:

$$\frac{dx}{dt} = ax\left(1 - \frac{x}{N}\right) \tag{3.2}$$

Here $a, N$ are positive parameters, $a$ gives the rate of population growth when $x$ is small and $N$ represent a sort of "ideal" population. Without loss of generality, we will assume that $N = 1$. That is, we will choose units so that carrying capacity is exactly 1 unit of population and $x(t)$ therefore present the fraction of ideal population present at time $t$. The logistic model reduces to

$$\frac{dx}{dt} = ax(1 - x) \tag{3.2'}$$

With the help of the Mathematica commands we can analyze the different system behaviors depending on the values of the growth parameter. For certain values of this parameter, the population settles to a fixed size over the years. This is called a fixed point of the system.

For $a = 3.83$, we consider the model $\frac{dx}{dt} = 3.83x(1 - x)$. By applying Mathematica constructed codes, the stability types of fixed points [5] are:

```
OneDimensionalStability[3.83*x*(1-x), x]
f has hyperbolic unstable fixed point at x=0.
f has hyperbolic unstable fixed point at x=0.738903
```

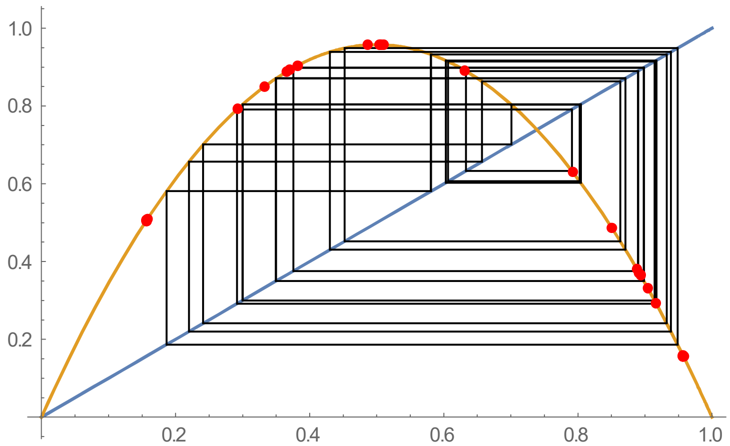


**Figure 2.** CobWeb Diagram

**The Cobweb Diagram**. [1,4] The method connects the successive points of an orbit on the graph of the function using as bridges the corresponding points on the line $y = x$ . The model has a stable periodic orbit of period 3. The red point on the graph represent the periodic points.

We want to plot the sequence $(n, x_n)$ to visualize the state of the orbit as a function of $n$ . We compress $n$ dividing it by 10. Hence 10 states of the orbit are condensed on every interval of length 1. We choose $f$ , an initial condition $x$ , and a starting value for $n$ . For the same initial condition $x_0 = 0.3$ , we will varies with parameter $a$ to see the behavior of the model.

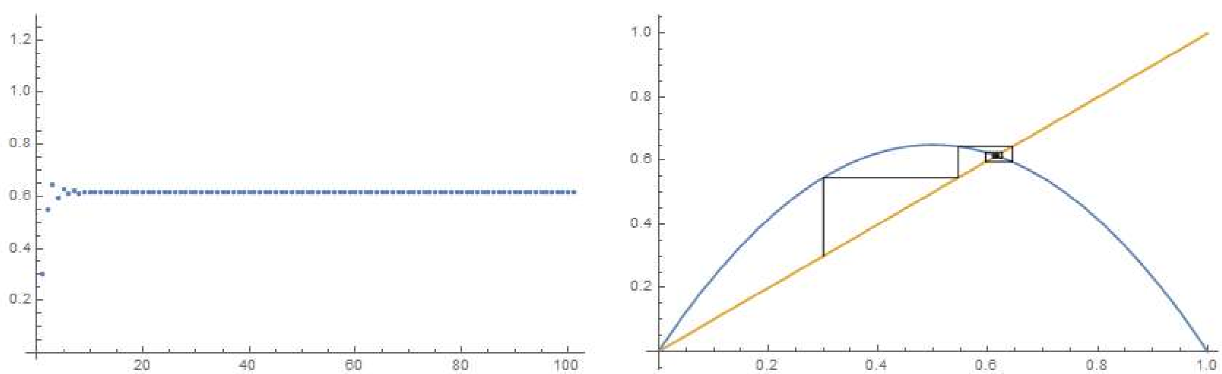


**Figure 3.** Time series (left) and first return map (right) of the fixed point logistic map with $a = 2.6$

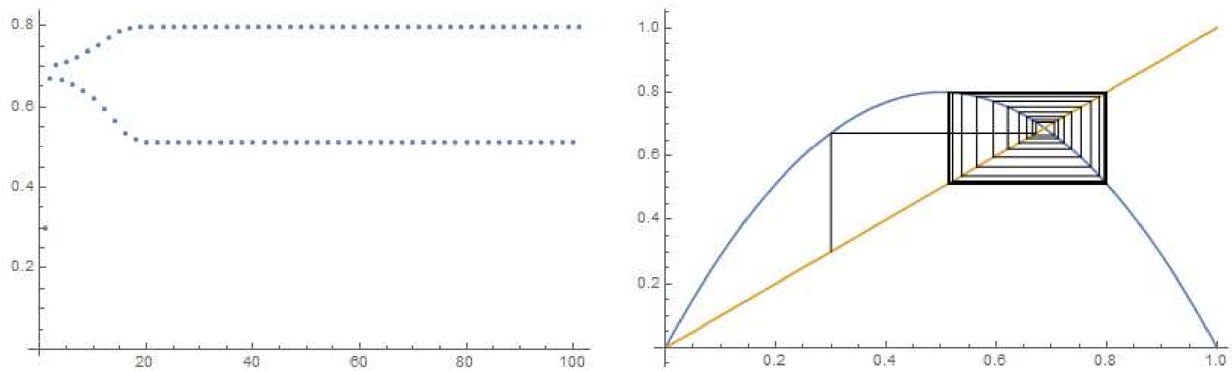


**Figure 4.** Time series (left) and first return map (right) of the fixed point logistic map with $a = 3.2$

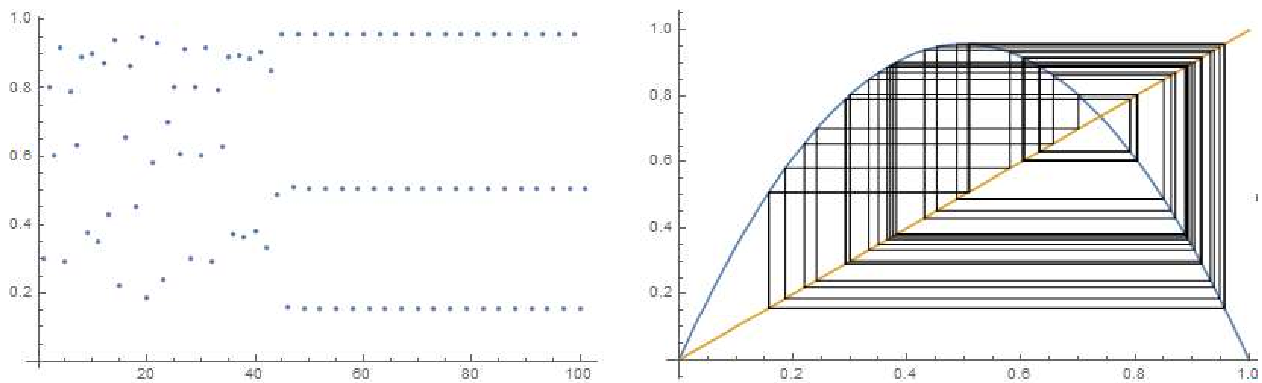


**Figure 5.** Time series (left) and first return map (right) of the fixed point logistic map with $a = 3.83$

With the help of these commands we can analyze the different system behaviors depending on the values of the growth parameter. For certain values of this parameter, the population settles to a fixed size over the years. This is called a fixed point of the system (Fig.3). When the parameter value is increased, the system jumps back and forth between two different points [2,4]. This is called a period-2 orbit (Fig.4). In addition, the system may also evolve under an infinite number of points in a random-looking form (Fig.5). This behavior is known as deterministic chaos, since seemingly stochastic (chaotic) behavior is obtained from the dynamics of a deterministic system.

*Example 3.3. (Applications to Chemical Kinetics)*

The chemical equation for the reaction between nitrous oxide and oxygen to form nitrogen dioxide at $25^\circ C$, $2NO + O_2 \to 2NO_2$ obeys the law of mass action [3]. The rate equation is given by $\frac{dx}{dt} = k(a-x)^2\left(b - \frac{x}{2}\right)$ where $x = [NO_2]$ is the concentration of nitrogen dioxide $k$ is the rate constant, $a$ is the initial concentration of $NO$, and $b$ is the initial concentration of $O_2$.

The concentration of nitrogen dioxide, after time $t$ given that $k = 0.00713 l^2 M^{-2} s^{-1}$ , $a = 4Ml^{-1}$, $b = 1Ml^{-1}$ and $x(0) = 0Ml^{-1}$.

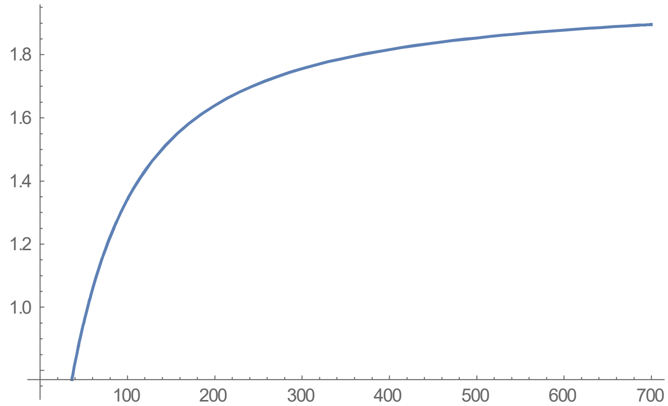


**Figure 6.** *The Concentration of* $NO_2$ *in moles per liter against time in seconds using time series*

Investigation of stability using Mathematica coding [4] to find fixed points and their nature, give those results:

```
OneDimensionalStability[.00713 * (4 - x) ^2 * (1 - x / 2), x]
f has hyperbolic stable fixed point at x=0.102733
```

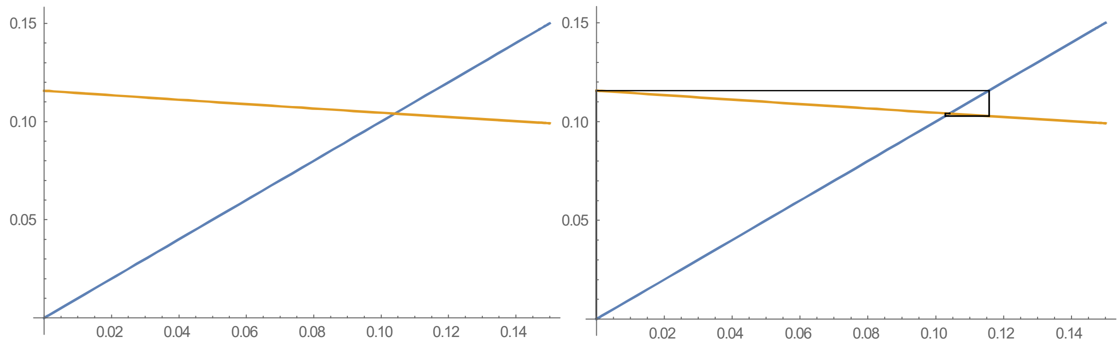


**Figure 7.** *CobWeb Diagram*

## 4. Mathematica Codes

- *Finding fixed point and their stability (Example 3.1, Example 3.2, Example 3.3)*

```
OneDimensionalStability[g_, x_] := Module[{G, u, l, czl, czl1, Sch},
  G[u_] := g /. x → u; czl = Solve[G[u] == u, u]; czl1 = u /. czl; l = Length[czl1];
  Sch[u_] := G'''[u] / G'[u] - (3 / 2) * (G''[u] / G'[u]) ^ 2;
  Do[Which[
    -1 < G'[czl1[[i]]] < 1, Print["f has hyperbolic stable fixed point at ", "x=", czl1[[i]]],
    G'[czl1[[i]]] > 1, Print["f has hyperbolic unstable fixed point at ", "x=", czl1[[i]]],
    G'[czl1[[i]]] < -1, Print["f has hyperbolic unstable fixed point at ", "x=", czl1[[i]]],
    G'[czl1[[i]]] == 1, If[G''[czl1[[i]]] ≠ 0,
     Print["f has nonhyperbolic unstable (semistable) fixed point at ", "x=", czl1[[i]]],
     Which[G'''[czl1[[i]]] < 0, Print["f has nonhyperbolic stable point at ", "x=", czl1[[i]]],
      G'''[czl1[[i]]] > 0, Print["f has nonhyperbolic unstable point at ", "x=", czl1[[i]]]]],
    G'[czl1[[i]]] == -1, Which[Sch[czl1[[i]]] < 0,
     Print["f has nonhyperbolic stable fixed point at ", "x=", czl1[[i]]], Sch[czl1[[i]]] > 0,
     Print["f has nonhyperbolic unstable fixed point at ", "x=" czl1[[i]]]]]], {i, 1, l}]]
```

- **Time Series and CobWeb diagram (Example 3.3, Fig.3, Fig. 4, Fig. 5)**

```
Logistic[a_] := Function[x, a * x * (1 - x)];

Orbit[map_, x0_, n_] := NestList[map, x0, n];
IterativeProcess[map_, x0_, {min_, max_}] := Module[{fr, orb},
  orb = Orbit[map, x0, 50];
  fr = MapThread[Line[{{#1, #1}, {#1, #2}, {#2, #2}}] &,
    {Drop[orb, -1], Drop[orb, 1]}];
  Show[Plot[{map[x], x}, {x, min, max}], Graphics[{fr}]]]


Show[GraphicsArray[{ListPlot[Orbit[Logistic[3.2], 0.3, 100]],
   IterativeProcess[Logistic[3.2], 0.3, {0, 1}]}]]
```

- *CobWeb Diagram and Periodic Points (Example 3.2, Fig. 2)*

```
f[x_] := 3.83 * x * (1 - x)
q1 = Plot[{x, x = f[x]}, {x, 0, 1}]
x = .3
0.3
q2 = Graphics[{Line[Flatten[Table[{{x, x}, {x, x = f[x]}}, {30}], 1]]}]
Show[q1, q2]
q3 = Graphics[{RGBColor[1, 0, 0], PointSize[0.015], Table[Point[{x, x = f[x]}], {30}]}]
Show[q1, q2, q3]
```

- *Investigating Stability and phase portrait of general logistic model varying by parameter (Example 3.1, Fig 1.)*

```
Manipulate[Module[{T = 100, sol, x, t},
  sol = First[x /.
     NDSolve[{x'[t] == μ * x[t] * (1 - x[t - 1]), x[t /; t ≤ 0] == .3}, x, {t, 0, T}]];
  If[pp, ParametricPlot[{sol[t], sol[t - 1]}, {t, 1, T}, PlotRange → {{0, 3}, {0, 3}}],
   Plot[sol[t], {t, 0, T}, PlotRange → {{0, 100}, {0, 3}}]]],
 {{pp, False, "Phase Portrait"}, {False, True}}, {{μ, 1}, 0, 4}]
```

## References


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